Homework 11, Math 4121, due 17, April 2014

(1) For $0 < r < p < s < \infty$, prove that $L^r \cap L^s \subset L^p$. Further, if $\mu(X) < \infty$, prove that $L^s \subset L^r$ if $0 < r < s < \infty$.

(2) If $f, g$ are positive measurable functions on $X$ with $\mu(X) = 1$ such that $f(x)g(x) \geq 1$ for all $x \in X$, prove that $\int_X f d\mu \int_X g d\mu \geq 1$.

(3) Let $X = (0, \infty)$ and let $f \in C_c(X)$ which is positive. Define $F(x) = \frac{1}{x} \int_0^x f(t) dt$ for $x \in X$. Prove that $F \in L^p(X)$ for any $1 < p < \infty$ and $\|F\|_p \leq \frac{p}{p-1} \|f\|_p$. (Same is true for any $f \in L^p(X)$.)

(4) Suppose $\mu(X) = 1$ and $f : X \to [0, \infty]$ a measurable function. Let $A = \int_X f d\mu$. Then prove that, $\sqrt{1 + A^2} \leq \int_X \sqrt{1 + f^2} d\mu \leq 1 + A$. If $X = (0,1)$ with the Lebesgue measure and $f = g'$ for a differentiable function, this must be familiar to you from calculus.

(5) Let $p,q$ be conjugate with $1 < p, q < \infty$. For any $f \in L^p$, we have a map $T_f : L^q \to \mathbb{R}$ defined as $T_f(g) = \int_X f g d\mu$.

(a) If $L : L^q \to \mathbb{R}$ is a linear functional, prove that $L$ is continuous if and only if there exists a non-negative number $l$ such that $|L(g)| \leq l \|g\|_q$ for all $g$.

(b) Prove that $|T_f(g)| \leq \|f\|_p \|g\|_q$. Deduce that $T_f$ is a continuous linear functional.

(c) Denoting by $H$ the set of all continuous linear functionals on $L^q$, for any $L \in H$, define $\|L\|_H = \inf\{l \geq 0 : |L(g)| \leq l \|g\|_q\}$ and prove that this makes $H$ a normed linear space. (The last phrase just means $H$ is a vector space and the norm has all the basic properties of a norm in $L^p$-spaces.)

(d) Prove that the map $A : L^p \to H$ defined by $f \mapsto T_f$ is continuous linear with $\|f\|_p = \|T_f\|_H$ for all $f$. (In fact, this map is a bijection.)