Homework 3, Math 4121, due 6 Feb 2014

1) Prove that if \( f_n \to f, g_n \to g \) uniformly on a set \( S \), so does \( f_n + g_n \to f + g \). Give an example where \( f_n g_n \) does not converge uniformly to \( fg \).

2) Let \( \{f_n\} \) be a sequence of continuous functions on a compact interval monotonically decreasing (or increasing), converging pointwise to a continuous function \( f \). Prove that the convergence is uniform.

Deduce using the above, if \( \{f_n\} \) is a sequence of positive continuous functions on a compact interval and \( \sum_{n=1}^{\infty} f_n \) converges pointwise to a continuous function, then the convergence is uniform.

3) Let \( X \subset \mathbb{R}^n \) be a closed subset and \( U \) an open subset containing \( X \). Given a continuous function on \( X \), show that it has an extension to all of \( \mathbb{R}^n \), which is continuous and zero outside \( U \).

4) For any \( k \in \mathbb{N} \cup \{0\} \) show that the series \( \sum_{n=1}^{\infty} n^k x^n \) has a radius of convergence 1. Denoting this function by \( f_k(x) \) in \((-1, 1)\), calculate \( f_2(\frac{1}{2}) \).

5) Let \( J(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n (n!)^2} \) (Bessel function of order zero). Find its radius of convergence, and show that it is a solution of the differential equation \( xy'' + y' + xy = 0 \).

6) (a) Assume \( \sum_{n=0}^{\infty} a_n x^n \) has a radius of convergence \( R > 0 \). Further assume that \( \sum_{n=0}^{\infty} a_n R^n \) also converges. Prove that \( \sum_{n=0}^{\infty} a_n x^n \) converges uniformly on \([0, R]\).

(b) Let \( \lim_{n \to \infty} a_n = L \) and \( f(x) \) be the series \( \sum a_n x^n \). Show that \( \lim_{x \uparrow 1}(1 - x)f(x) = L \).