Homework 8, Math 4121, due 20, March 2014

In the following, \((X, \mathcal{F}, \mu)\) will be a measure space.

(1) Let \(f_1 \geq f_2 \geq \cdots \geq f \geq 0\) be a sequence of measurable functions with range \([0, \infty]\) and \(\lim f_n = f\). Assume that \(f_1 \in L^1(\mu)\). Then prove that \(\lim \int_X f_n d\mu = \int_X f d\mu\). Give an example to show that the condition \(f_1 \in L^1(\mu)\) is necessary.

(2) Let \(A\) be a measurable set and consider the sequence of functions, \(f_n = \chi_A\) if \(n\) is odd and \(f_n = 1 - \chi_A\) is \(n\) is even. Use this to deduce that strict inequality can occur in Fatou’s lemma.

(3) Suppose \(\mu(X) < \infty\) and \(f_n : X \to \mathbb{R}\) is a sequence of bounded (that is, \(|f_n| < M_n\) for constants \(M_n\)) measurable functions, converging uniformly to \(f\). Prove that \(\lim \int_X f_n d\mu = \int_X f d\mu\).

(4) Let \(f : X \to [0, \infty]\) be measurable and let \(A_1, A_2, \ldots\) be a countable collection of pairwise disjoint measurable sets and let \(A = \bigcup A_n\). Prove that \(\int_A f d\mu = \sum_{n=1}^{\infty} \int_{A_n} f d\mu\).

(5) Let \(f \in L^1(\mu)\). Prove that, given \(\epsilon > 0\), there exists a \(\delta > 0\) such that if \(A \in \mathcal{F}\) with \(\mu(A) < \delta\) then \(\int_A |f| d\mu < \epsilon\).

(6) Let \(f_n : X \to \mathbb{R}\) be a sequence of measurable functions. Prove that the set of points where \(\{f_n\}\) converge is a measurable set.