See 1.1 Exer. 12

\[ x_1 - 3x_2 + 4x_3 = -4 \quad \text{(1)} \]
\[ 3x_1 - 7x_2 + 7x_3 = -8 \quad \text{(2)} \]
\[ -4x_1 + 6x_2 - x_3 = 7 \quad \text{(3)} \]

We try to get rid of \( x_1 \) from (2) & (3) by row operations.

Since coefficient of \( x_1 \) in (2) is 3, and \( 1 \) in (1), we multiply (1) by 3 and subtract from (2)

\[ 3x_1 - 7x_2 + 7x_3 = -8 \quad \text{(2)} \]
\[ 3x_1 - 9x_2 + 12x_3 = -12 \quad 3 \times (1) \]
\[ 0 \cdot x_1 + 2x_2 - 5x_3 = 4 \quad (2) - 3 \times (0) \]

Similarly,
\[-4x_1 + 6x_2 - x_3 = 7 \quad (3)\]
\[4x_1 - 12x_2 + 16x_3 = -16 \quad 4 \times (1)\]
\[0x_1 - 6x_2 + 15x_3 = -9 \quad (3) + 4 \times (1)\]

Thus the 3 equations become:
\[2x_2 - 5x_3 = 4 \quad (2')\]
\[-6x_2 + 15x_3 = -9 \quad (3')\]

This system is equivalent to our original system.

(2') and (3') involve only \(x_1\).

So we use similar row operations using (2') & (3') and eliminate \(x_2\) from (3')
\[-6x_2 + 15x_3 = -9 \quad (3')\]
\[6x_2 - 15x_3 = 12 \quad (2') \times 3\]
\[
6x_2 - 15x_3 = 12 \quad (2') \times 3
\]
\[
0x_2 + 0x_3 = 3 \quad (3) + 3x(2')
\]
So we have an equivalent system:
\[
x_1 - 3x_2 + 4x_3 = -4
\]
\[
2x_2 - 5x_3 = 4
\]
\[
0 = 3
\]

The last equation is never true, so our system has no solutions, i.e., inconsistent.

Exer 14:
\[
x_1 - 3x_2 = 5 \quad (1)
\]
\[
-x_1 + x_2 + 5x_3 = 2 \quad (2)
\]
\[
x_2 + x_3 = 0 \quad (3)
\]

Replace (2) by (2) + (1). (Do you see why?) We get an
equivalent system:

\[ x_1 - 3x_2 = 5 \]
\[ -2x_2 + 5x_3 = 7 \]
\[ x_2 + x_3 = 0. \]

For arithmetic ease, we flip \( 2 \) & \( 3 \):

\[ x_1 - 3x_2 = 5 \]
\[ -2x_2 + 5x_3 = 7. \]
\[ x_2 + x_3 = 0. \]

Last equation gives

\[ x_3 = 1 \]

so \( 0.1 \) lines in \( \Theta \), we get \( \ldots \)
Substituting in (2), we get

\[ x_2 + 1 = 0 \quad \text{or} \quad x_2 = -1 \]

Substituting in (1) gives

\[ x_1 + 3 = 5 \quad \text{or} \quad x_1 = 2 \]

So our unique solution is,

\[ x_1 = 2, \quad x_2 = -1, \quad x_3 = 1 \]

(You must check this is indeed a solution to our original system of equations.)

---

Exes 27:

\[ x_1 + 3x_2 = f \quad - \quad (1) \]

\[ cx_1 + dx_2 = g \quad - \quad (2) \]

We do the row operation,

(2) \(-c \times (1)\) to get,

\[ x_1 + 3x_2 = f \]
\[(d-3c)x_2 = g - 3f\]

Since \(f, g\) can be arbitrary, \(g - 3f\) can be non-zero. Then the last equation has a solution only if \(d - 3c \neq 0\). (Remember this from the first class?)

So \(d - 3c \neq 0\) is a necessary condition for this system to be consistent for any choice of \(f\) and \(g\).

I claim this is also sufficient. Then, we have from the last equation:

\[x_2 = \frac{g - 3f}{d - 3c}\]

Substitute in \(x_2\) to get.
\[ x_1 + 3 \cdot \frac{9-3f}{d-3c} = f \]

or
\[ x_1 = f - 3 \cdot \frac{9-3f}{d-3c} \]

This \(d-3c \neq 0\) is necessary and sufficient for the system to be consistent for any choice of \(f\) and \(g\).

See 1.2 Exer 4:

\[
\begin{bmatrix}
1 & 3 & 5 & 7 \\
3 & 5 & 7 & 9 \\
5 & 7 & 9 & 1
\end{bmatrix}
\]

Subtract \(3 \times 1^{st}\) row from \(2^{nd}\) row

and \(5 \times 1^{st}\) row from \(3^{rd}\) row:

(\text{So the first column is the pivot column. \(a_1^t = 1\) is the pivot.})

\[
\begin{bmatrix}
1 & 3 & 5 & 7 \\
0 & -4 & -8 & -12 \\
0 & -8 & -16 & -34
\end{bmatrix}
\]
Now we "hide" the first row and then the second column is the pivot column (first non-zero column.) Then we can use -4 as the pivot. So subtract 2x second row from the 3rd to get:

\[
\begin{bmatrix}
1 & 3 & 5 & 7 \\
0 & -4 & -8 & -12 \\
0 & 0 & 0 & -10
\end{bmatrix}
\]

Now hide 1st & 2nd row, and the last row has 4 column as the pivot column.

This is the echelon form. To get to the reduced echelon form we do reverse row operations.

We multiply the last row
by \(-\frac{1}{10}\) to get
\[
\begin{bmatrix}
1 & 3 & 5 & 7 \\
0 & -4 & -8 & -12 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
Next we subtract \(7 \times 3\) row from 1st row, add \(12 \times 3\) row to the 2nd row to get
\[
\begin{bmatrix}
1 & 3 & 5 & 0 \\
0 & -4 & -8 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
Next, the pivot is \(-4\), so multiply by \(-\frac{1}{4}\) the second row to get
\[
\begin{bmatrix}
1 & 3 & 5 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
Next use the pivot to get the position above zero. So we
multiply 2\textsuperscript{nd} row by 3 and subtract from row 1. To get:

\[
\begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

This is the reduced echelon form and the circled positions are the pivots.

\[\times\]

Exercise 8

\[
\begin{bmatrix}
1 & 4 & 0 & 7 \\
2 & 7 & 0 & 10
\end{bmatrix}
\]

The first column is pivot column and the top entry 1 in \(\begin{bmatrix}1' \end{bmatrix}\) is non-zero, so can be used as the pivot. So we multiply 1\textsuperscript{st} row
by 2 and subtract from 2nd row:

\[
\begin{bmatrix}
1 & 4 & 0 & 7 \\
0 & -1 & 0 & -4
\end{bmatrix}
\]

We can use this to solve, but let us convert this to reduced echelon form.

The pivot once you hide the first row is -1, in the second column. Divide 2nd row by -1 to get:

\[
\begin{bmatrix}
1 & 4 & 0 & 7 \\
0 & 1 & 0 & 4
\end{bmatrix}
\]

Now subtract 4x 2nd row from 1st row:

\[
\begin{bmatrix}
1 & 0 & 0 & -9 \\
0 & 1 & 0 & 4
\end{bmatrix}
\]

Writing as equations:

\[x_1 = -9\]
\[ x_1 = -1 \]
\[ x_2 = 4 \]

So \( x_3 \) is a free variable and so a solution is of the form \((-9, 4, x_3)\), where \( x_3 \) can be any number.