ANSWERS TO QUIZ 2

Show your work not just your final answer

(1) Define two matrices

$$A = \begin{pmatrix} 5 & 2 \\ -1 & 3 \end{pmatrix}; \qquad B = \begin{pmatrix} 2 & 1 \\ -4 & 0 \end{pmatrix};$$

Compute $AB - BA$.

Solution:

$$AB = \begin{pmatrix} 2 & 5 \\ -14 & 1 \end{pmatrix}; \quad BA = \begin{pmatrix} 9 & 7 \\ -20 & 8 \end{pmatrix};$$

$$\boxed{AB - BA = \begin{pmatrix} -7 & -2 \\ 6 & 7 \end{pmatrix}}$$

(2) Consider the 1×4 matrix,

$$C = \begin{bmatrix} 1 & 2 & -1 & -2 \end{bmatrix}$$

Compute CC^T.

Solution:

$$CC^T = \begin{bmatrix} 1 & 2 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix} = 1 + 4 + 1 + 4 = \boxed{10}$$

(3) Find the inverse of the following matrix, if it exists:

$$\begin{bmatrix} 2 & 7 \\ 3 & 11 \end{bmatrix}$$

Solution: The inverse exists because the deteminant is nonzero: $ad - bc = (2)(11) - (7)(3) = 1$.

Use the formula for inverses of 2×2 matrices:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Thus

$$\boxed{\begin{bmatrix} 2 & 7 \\ 3 & 11 \end{bmatrix}^{-1} = \begin{bmatrix} 11 & -7 \\ -3 & 2 \end{bmatrix}}$$

(4) Find an invertible 2×2 matrix A such that $A + A^T$ is singular.

Solution: Many solutions exist. Perhaps the simplest is

$$\boxed{A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}$$

which has two pivot columns hence is invertible, but for which $A + A^T = 0$ has all of $\mathbf{R}^2$ as a nontrivial nullspace and thus is singular.

(5) For what value of k is the following matrix singular:

$$\begin{bmatrix} 2 & 8 \\ k & -7 \end{bmatrix}$$

Solution: The matrix is singular if and only the determinant is 0, so solve for k in

$$0 = ad - bc = (2)(-7) - (8)(k) = -14 - 8k \iff \boxed{k = -7/4}$$

(6) The 2×2 elementary matrix E can be obtained from the identity using the row operation $R_2 = R_2 + 3R_1$. Find EA if

$$A = \begin{bmatrix} -8 & -1 \\ 1 & 8 \end{bmatrix}$$

Solution: First find E:

$$E = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

Next, compute EA:

$$EA = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -8 & -1 \\ 1 & 8 \end{bmatrix} \implies \boxed{EA = \begin{bmatrix} -8 & -1 \\ -23 & 5 \end{bmatrix}}$$

Equivalently, recognize that multiplication by E on the left adds 3 times row 1 into row 2 of A.

(7) Find the LU factorization of the following matrix. No row interchanges should be made.

$$A = \begin{bmatrix} 2 & -2 & -1 \\ 8 & -9 & -6 \\ 10 & -7 & 5 \end{bmatrix}$$

Solution: The first column of L is obtained by dividing the first column of A by the diagonal element $a_{11} = 2$, and the partially row reduced matrix $A^{(1)}$ is obtained by subtracting the respective multiples of row 1 of A from rows 2 and 3:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & * & 1 \end{bmatrix} \qquad A^{(1)} = \begin{bmatrix} 2 & -2 & -1 \\ 0 & -1 & -2 \\ 0 & 3 & 10 \end{bmatrix}$$

The second column of L is obtained by dividing the second column of $A^{(1)}$ by the diagonal element $a_{22} = -1$, and the upper triangular matrix $U = A^{(2)}$ is obtained by subtracting the resulting multiple of row 2 of $A^{(1)}$ from row 3:

$$\boxed{L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & -3 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & -2 & -1 \\ 0 & -1 & -2 \\ 0 & 0 & 4 \end{bmatrix}}$$

NOTE: it is wise to check your work by multiplying LU and comparing with A.

(8) Use the following LU factorization to find all solutions to $A\mathbf{x} = \mathbf{b}$:

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & -1 & 1 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 0 & -9 \\ 0 & 0 \end{bmatrix}; \qquad \mathbf{b} = \begin{bmatrix} -42 \\ -189 \\ -147 \end{bmatrix}.$$

Solution: First solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$:

$$L\mathbf{y} = \begin{bmatrix} y_1 \\ 3y_1 + y_2 \\ 5y_1 - y_2 + y_3 \end{bmatrix} = \mathbf{b} = \begin{bmatrix} -42 \\ -189 \\ -147 \end{bmatrix}.$$

This gives

$y_1 = b_1 = -42,$

$y_2 = b_2 - 3y_1 = -189 - (3)(-42) = -63,$

and

$y_3 = b_3 + y_2 - 5y_1 = -147 + (-63) - (5)(-42) = 0.$

Next, solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$:

$$U\mathbf{x} = \begin{bmatrix} 4x_1 - 2x_2 \\ -9x_2 \\ 0 \end{bmatrix} = \mathbf{y} = \begin{bmatrix} -42 \\ -63 \\ 0 \end{bmatrix}.$$

This is a consistent system which we may solve with
$x_2 = (-63)/(-9) = 7$ and
$x_1 = (-42 + 2x_2)/(4) = (-42 + (2)(7))/(4) = -7.$

Hence the unique solution to $A\mathbf{x} = \mathbf{b}$ is

$$\boxed{\mathbf{x} = \begin{bmatrix} -7 \\ 7 \end{bmatrix}.}$$

(9) Find the rank and nullity of the following matrix:

$$A = \begin{bmatrix} 2 & -6 & -4 & 1 & 2 \\ 1 & -3 & -3 & -2 & 2 \\ -1 & 3 & 2 & 0 & 0 \end{bmatrix}$$

Solution: Row reduce A to find the number of pivot columns. It is convenient to first interchange rows 1 and 2:

$$\begin{bmatrix} 1 & -3 & -3 & -2 & 2 \\ 2 & -6 & -4 & 1 & 2 \\ -1 & 3 & 2 & 0 & 0 \end{bmatrix}$$

Second, replace $R_2 \leftarrow R_2 - 2R_1$ and $R_3 \leftarrow R_3 + R_1$:

$$\begin{bmatrix} 1 & -3 & -3 & -2 & 2 \\ 0 & 0 & 2 & 5 & -2 \\ 0 & 0 & -1 & -2 & 2 \end{bmatrix}$$

Third, replace $R_2 \leftarrow R_2 + 2R_3$:

$$\begin{bmatrix} 1 & -3 & -3 & -2 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & -2 & 2 \end{bmatrix}$$

Finally, multiply $R_3 \leftarrow -R_3$ and interchange $R_2 \leftrightarrow R_3$:

$$\begin{bmatrix} 1 & -3 & -3 & -2 & 2 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

This results in echelon form for A. Three columns (1, 3, and 4) are pivot columns, and two columns (2 and 5) are free columns. Thus,

$$\boxed{\text{rank}(A) = 3, \qquad \text{nullity}(A) = 2}$$

(10) Let A be a 12×17 matrix with rank 5. Find the nullity of A.

Solution: Use the rank+nullity theorem. Rank(A)+nullity(A)=17, the number of columns of A, so 5+nullity(A)=17, so

$$\boxed{\text{nullity}(A) = 12.}$$

(11) Find the determinant of the following matrix:

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 2 & -6 & -4 & 0 & 0 \\ 1 & -3 & -3 & -1 & 0 \\ -1 & 5 & 12 & 0 & 3 \end{bmatrix}$$

Solution: The determinant of an upper triangular matrix such as this A is the product of the diagonal elements, so

$$\boxed{\det A = (2)(1)(-4)(-1)(3) = 24.}$$

(12) Use expansion by minors to find the determinant of the following matrix:

$$A = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & -6 & 0 & 0 \\ -1 & 5 & 0 & 3 \end{bmatrix}$$

Solution: This may be done in three steps.
First expand using column 4:

$$\det A = (3) \det \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 0 \\ 2 & -6 & 0 \end{bmatrix}$$

Second, expand using column 3 of the 3×3 minor:

$$\det A = (3)\det\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 0 \\ 2 & -6 & 0 \end{bmatrix} = (3)(1)\det\begin{bmatrix} 2 & 1 \\ 2 & -6 \end{bmatrix}$$

Finally, use the formula $ad - bc$ for the remaining 2×2 minor:

$$\boxed{\det A = (3)(1)[(2)(-6) - (1)(2)] = -42}$$