ANSWERS TO MIDTERM 1

Show your work not just your final answer

(1) Decide whether the following statements are True or False. (You must explain your answers, by either quoting results from the book or supplying your reasons)

(a) Every matrix is row equivalent to a unique matrix in reduced echelon form.
True. This is Theorem 1, in section 1.2.

(b) If a system $A\mathbf{x} = \mathbf{b}$ has more than one solution, then so does the system $A\mathbf{x} = \mathbf{0}$.
True. If $\mathbf{x} \neq \mathbf{y}$ are two distinct solutions of the above equation, then $A(\mathbf{x} - \mathbf{y}) = A\mathbf{b} - A\mathbf{b} = \mathbf{0}$. Thus, $\mathbf{0}$ and $\mathbf{x} - \mathbf{y} \neq \mathbf{0}$ are both solutions of $A\mathbf{x} = \mathbf{0}$.

(c) If A, B are row equivalent $m \times n$ matrices and if the columns span $\mathbb{R}^m$, then so do the columns of B.
True. See Theorem 4 in section 1.4.

(2) Determine h, k such that the solution set of the system,

$$\begin{array}{ccccc} x & + & 3y & = & k \\ 4x & + & hy & = & 8 \end{array}$$

is a) is empty, b) contains a unique solution and c) contains infinitely many solutions.

The augmented matrix is

$$\begin{bmatrix} 1 & 3 & k \\ 4 & h & 8 \end{bmatrix}$$

By subtracting 4 times the first row from the second, we get,

$$\begin{bmatrix} 1 & 3 & k \\ 0 & h-12 & 8-4k \end{bmatrix}$$

The last equation says that if $h = 12$, but $k \neq 2$, the the system is inconsistent and thus have no solutions.

If $h = 12$ and $k = 2$, then we have a free variable and thus the system has infinitely many solutions.

Finally, if $h \neq 12$, we have a unique solution.

(3) Find a, b with $a^2 + b^2 = 1$ and satisfying the following equation.

$$\begin{bmatrix} a & 0 & -b \\ 0 & 1 & 0 \\ b & 0 & a \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2\sqrt{5} \\ 3 \\ 0 \end{bmatrix}$$

Multiplying the left side, we get two equations,

$$\begin{array}{rcrcl} 2a & - & 4b & = & 2\sqrt{5} \\ 4a & + & 2b & = & 0 \end{array}$$

The last equation gives $b = -2a$. Back-substituting for b in the first equation, one gets $2a + 10a = 2\sqrt{5}$ which gives $a = \frac{1}{\sqrt{5}}$ and thus $b = -\frac{2}{\sqrt{5}}$. Clearly $a^2 + b^2 = 1$, which we never used.

(4) Calculate the determinant of the matrix,

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix}$$

Show that the determinant is zero if and only if $x_i = x_j$ for some $i \neq j$.

We expand using the first column and thus the determinant is,

$$\begin{vmatrix} x_2 & x_2^2 \\ x_3 & x_3^2 \end{vmatrix} - \begin{vmatrix} x_1 & x_1^2 \\ x_3 & x_3^2 \end{vmatrix} + \begin{vmatrix} x_1 & x_1^2 \\ x_2 & x_2^2 \end{vmatrix}$$

$$= (x_2 x_3^2 - x_3 x_2^2) - (x_1 x_3^2 - x_3 x_1^2) + (x_1 x_2^2 - x_2 x_1^2)$$

$$= (x_1 - x_2)(x_2 - x_3)(x_3 - x_1)$$

The last says that the determinant is zero if and only if $x_i = x_j$ for some $i \neq j$.

(5) Decide whether the following statements are True or False.

(a) If A, B are $n \times n$ matrices with A invertible and $AB = BA$, then $A^{-1}B = BA^{-1}$.
True. If $AB = BA$, we have by multiplying both by A^{-1} on the left and right, $A^{-1}(AB)A^{-1} = A^{-1}(BA)A^{-1}$. That is, $BA^{-1} = A^{-1}B$ by using associativity.

(b) If A, B are $n \times n$ matrices, then $(A + B)(A - B) = A^2 - B^2$.

False. The left side, using distributivity, is $A^2 - AB + BA - B^2$ and since $AB \neq BA$ in general, it is not $A^2 - B^2$.

(c) If A is a 3×3 matrix and the equation $A\mathbf{x} = \mathbf{0}$ has a unique solution, then A is invertible.
True. $A\mathbf{x} = \mathbf{0}$ already has the trivial solution, $\mathbf{0}$. A is invertible is the same as the columns of A are linearly independent. If A is not invertible, the linear dependence of the column gives a non-zero $\mathbf{v}$ such that $A\mathbf{v} = 0$.

(6) Suppose,

$$AB = \begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix}, B = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}$$

Find A.

Notice that $\det B = (7 \times 1 - 3 \times 2) = 1$, so it is invertible and

$$B^{-1} = \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}$$

Thus,

$$A = (AB)B^{-1} = \begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix} = \begin{bmatrix} -3 & 13 \\ -8 & 27 \end{bmatrix}$$

(7) Decide whether the following statements are True or False.

(a) If A is a 3×3 matrix, $\det 5A = 5 \det A$
False. $\det 5A = 5^3 \det A$, since multiplication by 5 is same as multiplying by the 'scalar' matrix $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, whose determinant is 5^3.

(b) If $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ and if A is the matrix whose coluns are $\mathbf{u}$ and $\mathbf{v}$ and $\det A = 10$, then the area of the triangle with vertices $\mathbf{0}, \mathbf{u}$ and $\mathbf{v}$ is 10.
False. If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation corresponding to a matrix A and S is a region whose are is known, then the area of $T(S)$ is $\det A$ times the area of S. In our case, consider the transformation associated to A^T, then $T(\mathbf{e}_1) = \mathbf{u}, T(\mathbf{e}_2) = \mathbf{v}$. Thus the area in question is $T(\Delta)$, where, Δ is the triangle formed by $\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2$, whose are is $1/2$ and thus the area of $T(S) = \frac{1}{2} \det A^T = 5$.

(c) $\det A^T A \geq 0$, where A is a square matrix.
True. $\det A^T A = \det A^T \cdot \det A = \det A \cdot \det A = (\det A)^2 \geq 0$.

(d) If A is a square matrix with all its entries integers and $\det A = 1$, then A^{-1} has all its entries integers.
True. $A^{-1} = \frac{1}{\det A}\operatorname{adj}(A) = \operatorname{adj}(A)$. But the entries of adjugates are just the cofactors of A, which are determinants of matrices with integer entries up to sign and thus integers.

(8) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation determined by the matrix

$$A = \begin{bmatrix} a & 1 & 2 \\ 0 & b & 3 \\ 0 & 0 & c \end{bmatrix}$$

where a, b, c are positive numbers. Let S be the unit ball in $\mathbb{R}^3$ with center at the origin. i. e. the set of all points (x, y, z) with $x^2 + y^2 + z^2 \leq 1$. Use the fact that the volume of the S is $4\pi/3$ to calculate the volume of $T(S)$.

Volume of $T(S)$ is just $\det A$ times the volume of S. $\det A = abc$ and so volume of $T(S)$ is $4\pi abc/3$.