HOMEWORK 10

DUE NOVEMBER 21

(1) Let S be a finite set of cardinality n. We showed in class that $\operatorname{Aut} S = \{f : S \to S | f \text{a bijection}\}$ is a finite set with cardinality $n!$. If $f \in \operatorname{Aut} S$, clearly $f^r = f \circ f \circ \cdots \circ f$, r times, is also a bijection from S to S and thus $f^r \in \operatorname{Aut} S$. Show that for some $r \in \mathbb{N}$, f^r is the identity map.

(2) Let $f : \mathbb{N} \to S$ be a sequence where S is any set. If $g : \mathbb{N} \to \mathbb{N}$ is an *increasing* function (i. e. $g(n+1) > g(n)$ for all n), we call the sequence $f \circ g : \mathbb{N} \to S$ a *subsequence* of f.
   (a) Show that if f, g are as above, $g(n) \geq n$ for all $n \in \mathbb{N}$.
   (b) Now, let $S = \mathbb{Q}$. Show that if f gives a CS, any subsequence is also a CS. Also show that if $\{x_n\}$ is a CS, any subsequence is related (as defined in class) to $\{x_n\}$.
   (c) Give an example (with $S = \mathbb{Q}$) to show that a sequence may not be a CS, but a subsequence can be a CS.

(3) Let $\{x_n\}$ be a non-decreasing sequence (i. e. $x_{n+1} \geq x_n$ for all n). Assume the set $\{x_n | n \in \mathbb{N}\}$ is bounded above (i. e. there exists an M such that $x_n \leq M$ for all n). Show that $\{x_n\}$ is a CS.

(4) Let $0 \leq b_n \leq a_n$ for all n. Let $x_n = \sum_{i=1}^{n} a_i$ and $y_n = \sum_{i=1}^{n} b_i$. Show that if $\{x_n\}$ is a CS, so is $\{y_n\}$.

(5) Let $\{x_n\}$ be a CS with $x_n \geq r > 0$ for a fixed $r \in \mathbb{Q}$. Show that the sequence $\{\frac{1}{x_n}\}$ is also a CS. Give an example of a CS $\{x_n\}$ with $x_n > 0$ for all n, but $\{\frac{1}{x_n}\}$ is not a CS.