HOMEWORK 5

DUE OCTOBER 10

You may use any result that we proved in class and the ones I have posted for properties of numbers. If you can, quote them by name or statement or by some other identifying feature.

(1) Imitate the proof done in class to show that if $a > 0$ is a real number, there exists $\alpha > 0$, a real number such that $\alpha^2 = a$.

(2) Prove by induction $\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}$.

(3) Use the previous problem to show that the set $\{x_n | n \in \mathbb{N}\}$, where $x_n = \sum_{k=1}^{n} \frac{1}{k^2}$ of rational numbers is bounded above. Thus, this set has an lub, which we call $\alpha \in \mathbb{R}$. (Those of you in the writing class might recognize this as something do with π or $\zeta(2)$.) Show that given any positive real number ϵ, $\alpha - x_n < \epsilon$ for $n \gg 0$. (Again, taking a cue from your calculus classes, this is what one means by $\lim_{n\to\infty} x_n = \alpha$.)

(4) Let $x > 0$ be a real number (the positivity is not really necessary, but convenient) and let $N \in \mathbb{N}$ be such that $x < N$.
   (a) Show that there is a positive constant C such that for all $n \geq 2N$, $\frac{x^n}{n!} \leq C \frac{1}{2^{n-2N}}$.
   (b) Use the above to show that the sequence $z_n = \sum_{k=1}^{n} \frac{x^k}{k!}$ is bounded above.
   (c) Let α be the lub of the set $\{z_n | n \in N\}$. Show that $\lim_{n\to\infty} z_n = \alpha$.

   You will recognize from calculus that $\alpha = \exp x - 1$. One does not write e^x, since this notation suggests that we are exponentiating, which will turn out to be correct, though meaningless at present. We can write e^m for some $m \in \mathbb{Z}$ and using multiplication, this is meaningful, but multiplying e by itself x number of times, where $x \in \mathbb{R}$ has no meaning right now.

(5) Prove Pigeon Hole Principle using induction, as stated below.

   Let $P(n), n \in \mathbb{N}$ be the following statement: for any natural number $m > n$, if you distribute m balls in n boxes, then at least one box contains more than one ball.