HOMEWORK 7

DUE OCTOBER 31

If A, B are sets, we will use the notation $F(A, B)$ to denote the set of all functions from A to B.

(1) This is a preview of counting arguments we will encounter soon.
  (a) Let A be a set with exactly three elements and B be a set with exactly two elements. Calculate the number of elements in $F(A, B)$ and $F(B, A)$. (You do not have to tell me what these functions are, just the number of such distinct functions, but if you do not write them down for yourself, you would have lost a learning moment.)
  (b) If A has m elements and B has n elements, $m, n \in \mathbb{N}$, how many elements do $A \times B$ have?
  (c) If A has three elements, how many elements do $\mathcal{P}(A)$, the power set of A have?

(2) Let A be any set and let $B = \{0, 1\}$. Define a map $\phi : F(A, B) \to \mathcal{P}(A)$ by $\phi(f) = f^{-1}(0)$ for $f \in F(A, B)$. Show that ϕ is a bijection.

(3) For this problem, you may use your calculus knowledge.
  (a) Denote by $\mathcal{C}^0[0, 1]$ the set of all continuous functions on the closed interval $[0, 1]$. Define a map $\int : \mathcal{C}^0[0, 1] \to \mathbb{R}$ by, $\int(f) = \int_0^1 f$ for $f \in \mathcal{C}^0[0, 1]$. Is this map injective? Surjective?
  (b) Let $\mathcal{C}^0(\mathbb{R})$ denote the set of continuous functions on $\mathbb{R}$ and $\mathcal{C}^1(\mathbb{R})$, the set of functions on $\mathbb{R}$ which are differentiable with a continuous derivative. Then, consider the map $d : \mathcal{C}^1(\mathbb{R}) \to \mathcal{C}^0(\mathbb{R})$, given by $d(f) = f'$ for any $f \in \mathcal{C}^1(\mathbb{R})$, where as usual f' denotes the derivative. Is this map injective? Surjective?

(4) Let $f : A \to B, g : B \to C, h : C \to D$ be three functions, where A, B, C, D are sets. Show that $h \circ (g \circ f) = (h \circ g) \circ f$. Thus, the expression $h \circ g \circ f$ is unambiguous without the parenthesis.

(5) Let $\pi : A \to X$ be a surjective function and let $f : A \to Y$ be any function. Assume that for any $a, b \in A$, if $\pi(a) = \pi(b)$ then $f(a) = f(b)$. Show that there exists a unique function $g : X \to Y$ such that $f = g \circ \pi$.

(6) Verify the following are equivalence relations.
   (a) Let $A = \mathbb{N} \times \mathbb{N}$ and $(a, b) \sim (c, d)$ if $a + d = b + c$. (Can you identify the equivalence classes with $\mathbb{Z}$?)
   (b) Let $A = \mathbb{Z} \times \mathbb{Z}^*$ where $\mathbb{Z}^* = \mathbb{Z} - \{0\}$. The relation is given by, $(a, b) \sim (c, d)$ if $ad = bc$. (Can you identify the equivalence classes with $\mathbb{Q}$?)
   (c) The relation on $\mathbb{R}$ defined by $a \sim b$ if $a = b + n$ for some $n \in \mathbb{Z}$. (Can you identify the equivalence classes with the circle?)
   (d) Let $A = \mathbb{R}^2 - \{(0, 0)\}$ and the relation given by $(a, b) \sim (c, d)$ if for some $\lambda > 0$, real number, $(a, b) = (\lambda c, \lambda d)$. (Can you identify the equivalence classes with the circle?)