HOMEWORK 8

DUE NOVEMBER 7

(1) (a) Let $a \in \mathbb{N}$. Show that $a \leq \frac{a(a+1)}{2}$. Deduce that there exists a unique $\phi(a) \in \mathbb{N}$ such that $\frac{(\phi(a)-1)\phi(a)}{2} < a \leq \frac{\phi(a)(\phi(a)+1)}{2}$, defining a function $\phi : \mathbb{N} \to \mathbb{N}$.
   (b) Show that any $a \in \mathbb{N}$ is equal to $a = \frac{\phi(a)(\phi(a)+1)}{2} - j$ where $0 \leq j < \phi(a)$. Thus, we get a new function $\psi : \mathbb{N} \to \mathbb{N}$, by $\psi(a) = j$.
   (c) Define a function $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, by $f(a) = (\phi(a) - \psi(a), \psi(a) + 1)$. Show that this is a bijection.

(2) If $S \approx \mathbb{N}_n$ and $x \notin S$, show that $S \cup \{x\} \approx \mathbb{N}_{n+1}$.

(3) Show that if $S, T \subset \Omega$ are finite sets (Ω need not be finite), $\#(S \cup T) = \#S + \#T - \#(S \cap T)$.

(4) If $S \subset \mathbb{N}_n$, show that S is finite (by induction, for example) and $\#S \leq n$.

(5) If $f : S \to T$ is a surjective map with S finite, show that there exists a subset $S' \subset T$ such that $f_{|S'} : S' \to T$ is a bijection. Deduce that T is a finite set and $\#T \leq \#S$.

(6) Let S be a finite set with $\#S = n$.
   (a) Show that $\mathcal{P}(S)$ is a finite set and its cardinality is 2^n.
   (b) Let r be an integer with $0 \leq r \leq n$ and let $C_r(S) \subset \mathcal{P}(S)$ be the set of subsets A of S with $\#A = r$. Show that $\#C_r(S) = \binom{n}{r}$.
   (c) Deduce the formula $\sum_{r=0}^{n} \binom{n}{r} = 2^n$.

(7) If A, B are finite sets with $\#A = m, \#B = n$ at least one of m, n non-zero, show that $F(A, B)$ is a finite set and its cardinality is n^m.