THE IMPACT OF RIEMANN'S MAPPING THEOREM

GRANT OWEN

In the world of mathematics, scholars and academics have long sought to understand the work of Bernhard Riemann. Born in a humble German home, Riemann became one of the great mathematical minds of the 19^{th} century. Evidence of his genius is reflected in the greater mathematical community by their naming 72 different mathematical terms after him. His contributions range from mathematical topics such as trigonometric series, birational geometry of algebraic curves, and differential equations to fields in physics and philosophy [3]. One of his contributions to mathematics, the Riemann Mapping Theorem, is among his most famous and widely studied theorems. This theorem played a role in the advancement of several other topics, including Riemann surfaces, topology, and geometry. As a result of its widespread application, it is worth studying not only the theorem itself, but how Riemann derived it and its impact on the work of mathematicians since its publication in 1851 [3].

Before we begin to discover how he derived his famous mapping theorem, it is important to understand how Riemann's upbringing and education prepared him to make such a contribution in the world of mathematics. Prior to enrolling in university, Riemann was educated at home by his father and a tutor before enrolling in high school. While in school, Riemann did well in all subjects, which strengthened his knowledge of philosophy later in life, but was exceptional in mathematics. He enrolled at the University of Göttingen, where he learned from some of the best mathematicians in the world at that time. Professors like Moritz Stern, Carl Goldschmidt, and Carl Gauss inspired and tutored Riemann as his conceptions about mathematics were formed. Other notable mathematicians like Johann Listing and Lejeune Dirichlet also influenced Riemann's work. With regards to his mapping theorem, several of these scholars played a key role in Riemann's conceptual development of the theorem. Listing created the name Topology and influenced Riemann in his introduction to topologicial methods [3]. Gauss was Riemann's Ph.D advisor and focused Riemann's efforts on geometric thinking and proofs [3]. Finally Dirichlet, who was perhaps closest personally to Riemann, taught courses on the theory of numbers and analysis, which resulted in Riemann naming one of his mathematical principles after him, the Dirichlet principle [3], which would later be used in his mapping theorem. All of these mathematicians contributed to and helped Riemann develop his mathematical mind, which would ultimately advance mathematical thinking for years after his death.

Among his great contributions to mathematics, the Riemann mapping theorem provides a framework through which many mathematical models have followed. Developed as part of his Ph.D thesis in 1851 [3], it states that:

"Suppose that U is a connected open subset of $\mathbb{C}$ with $U \neq \mathbb{C}$. If U is holomorphically simply connected, then U is biholomorphic to the unit disc, i.e., there is a one-to-one holomorphic function from U onto the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$." [4]

A holomorphic function is a complex-valued function of one or more complex variables that is complex differentiable in a neighborhood of every point in its domain. If U is biholomorphic, then it is a holomorphic function where its inverse is also a holomorphic function. The theorem is noteworthy for two reasons.

(1) The methodology used in the proof of the theorem was difficult to carry out at the time and was not widely accepted until after other proofs were discovered years after Riemann's death. However, his method was mathematically sound and eventually proven correct.
(2) The theorem itself provides proof that any simply connected open subset of the plane is homeomorphic to any other [4].

While many mathematicians have applied this theorem in their work on the uniformization of Riemann surfaces and topology, along with various other topics, the greater challenge has been studying Riemann's proof and the conclusions that resulted from his work. It is a subject that continues to be studied today, as will be shown later.

Riemann's original proof used techniques and ideas that, while known in the mathematical community, were not widely used or practiced. Even today, his original proof is not the one most often used when justifying his theorem; however, every other proof developed since has stemmed from his original ideas of complex analysis. In attempting to prove his theorem, his general idea consists of constructing the biholomorphic map to the unit disc by solving an elliptic variational problem [4]. His original proof starts with showing that the function H on U is bounded, where U is the connected open subset of $\mathbb{C}$ as described in the original theorem. H is a biholomorphic mapping of U onto the unit disc D. He then proceeds to prove that $H : U \to D$ is biholomorphic and has $H(0) = 0$, such that $\frac{H(z)}{z}$ has a removable singularity at 0. H is defined as the biholomorphic map and D is the unit disc. Taking the antiderivative of $H(z)$, $L(z)$, and applying it to the expression $h(z)\exp(-L(Z))$ makes the expression constant. By changing $L(z)$ by an additive constant, we find $h(z)\exp(-L(Z)) = 1$ for all $z \in U$ [4].

A major point in Riemann's work is to consider the harmonic function $\mathrm{Re}L(z)$, which is equal to $\log|h(z)|$. This function, $\log|h(z)|$, must have a boundary value of $-\log|z_0|$ at z_0. This is important when considering the next step of the proof, the Dirichlet principle, named after his old mentor. This principle focuses on minimizing the integral

$$\int_U \left[\left(\frac{df}{dx}\right)^2 + \left(\frac{df}{dy}\right)^2\right] dxdy$$

with the condition that $f = g$ on the boundary ∂U of U. However, there is no reason to assume there is a minimum for this integral and thus no logical basis for applying this principle. For this reason, Riemann's mapping theorem was largely unproven for years until American mathematician William Osgood developed a proof that addressed this dilemma [4]. Osgood's proof allowed for mathematicians to then move beyond the Dirichlet principle and study Riemann's proof further. However, his methodology would be circumvented by an easier method to minimize the above integral, known as the Perron method. At the time, this method was not available to Osgood.

The Perron method did not appear for almost 70 years after Riemann proposed his theorem. While I do not fully comprehend the mathematics being used in this process, the description for this method comes from Robert Greene's publication *The Riemann Mapping Theorem from Riemann's Viewpoint* [4]. The theorem states:

"If U is a bounded connected open set in $\mathbb{C}$ with the property that for each boundary point ζ_0 of U, there is a weak barrier defined on an open disc around ζ_0, then, for any continuous function b on the boundary ∂U of U, the Perron upper envelope function P associated to b solves the Dirichlet problem on U with boundary values b, i.e., P is harmonic on U and $P \cup b$ is continuous on $U \cup \partial U$". [4]

This method supposes that a function P on U is a candidate for solution of the above integral, where $P(z) = \sup S(z)$, where sup is taken over all continuous subharmonic functions $S : U \to \mathbb{R}$. The method serves to find a general condition under which P has certain boundary values b. This condition is called a *barrier function*. Obtaining this *barrier function* guarantees that there is a harmonic function on any bounded holomorphically simple connected open set with h having the boundary value $-\log|z_o|$ for each $z_0 \in \partial U$ [4]. By knowing this harmonic function exists, mathematicians were then able to move beyond the Dirichlet problem and proceed with the proof.

Finally, the proof seeks to find a function $h(z)$, where $H(z) = zh(z)$ is a biholomorphic map to the unit disc. The function $h(0) \neq 0$ and $h(z)$ is also nonzero for every other $z \in U$ because $H(z)$ is supposed to be one-to-one and thus have a derivitive that does not disappear anywhere [4]. I will not include the math for how $h(z)$ is derived, but simply state that the equation $h(z) = \exp(g(z) + i\hat{g}(z))$ is derived to go along with $H(z) = zh(z)$[1]. These two equations then undergo an additional proof to prove that $H(z)$ is biholomorphic, which ultimately proves H to be one-to-one and onto the unit disc [4]. This technique is complicated and difficult to follow, but after more than 150 years of analysis and study, has been proven true.

Since Riemann's Mapping Theorem was first proposed, mathematicians have sought to find different proofs for the theorem. One such example is a proof published by the French mathematician Elisha Falbel. In her proof, she bases her argument around the topology of Stein manifolds and Hartog's Theorem in order to prove that Riemann's Mapping Theorem applies to domains with spherical boundaries [2]. A Stein manifold is a complex submanifold of the vector space of n complex dimensions. A submanifold is a subset S which has the structure of a manifold, and for which the inclusion map $S \to M$ satisfies certain properties, where M is the manifold under which S is a submanifold. A manifold is a topological space that locally resembles Euclidean space near each point. Hartog's Theorem is stated as:

"Let $\Omega \subset S$ be a domain in a Stein manifold of dimension $n \geq 2$ and $f : \Omega - K \to \mathbb{C}$ a holomorphic function, where $K \subset \Omega$ is compact and $\Omega - K$ connected. Then f admits an extension $F : \Omega \to \mathbb{C}$." [2]

By first generalizing and then proving this theorem, Falbel was able to provide a simple proof to Riemann's Mapping Theorem. This proof was published in 2003, 152 years after Riemann initially published his work at the University of Göttengen in 1851. After all these years, his

[1]For the derivation of this equation, see page 7 of *The Riemann Mapping Theorem from Riemann's Viewpoint*

original proof is still being analyzed to create new methods of proving his original theorem.

Another example of mathematicians branching off of Riemann's initial hypothesis and proof is Allen Weitsman's *A Counterexample to Uniqueness in the Riemann Mapping Theorem for Univalent Harmonic Mappings* [1]. In his work, Weitsman explored the proof of $H(z)$ being biholomorphic, which was discussed earlier [2]. He asserted that as a result of a theorem proven by other scholars in their own proof of Riemann's Mapping Theorem[3], there could be two univalent harmonic mappings onto the same domain D with the same normalization. By this argument, Weitsman only needed to provide a counterexample to the original assertion that there is only one univalent harmonic mapping onto the domain D; or, in other words, prove two possible harmonic mappings are on the same domain D. He did so successfully in his proof [1], which was published in 1999.

These are only two of many examples of Riemann's Mapping theorem being studied, explored, and expanded upon since its discovery by Bernhard Riemann. While his theorem itself carries various applications and continues to be explored in multiple topics such as Riemannian geometry and topology of surfaces, it is his proof that has driven mathematical research and study. Ultimately, Riemann's proof provided a method of the *"strictly topological fact that any simply connected open subset of the plane is homeomorphic to any other"* [4]. This proof laid the groundwork for further mathematical exploration into topics such as the use of elliptic equations in geometry, the analytic theory of Riemann surfaces via harmonic forms, and the Bochner technique for the circle of results [4]. While Riemann died shortly before his 40$^{\text{th}}$ birthday, his legacy in mathematics has carried on, thanks in part to the Riemann Mapping Theorem.

[2] See the first paragraph on page 4

[3] See footnote [2] of Weitsman's proof

References

[1] Allen Weitsman. *A Counterexample to Uniqueness in the Riemann Mapping Theorem for Univalent Harmonic Mappings.* Bulletin of the London Mathematical Society, 31.1 (1999):87-89. EBSCOhost. Web. 22 September 2017.

[2] Elisha Falbel. *A Simple Proof of the Riemann Mapping Theorem for Domains with Spherical Boundary* Complex Variables, 48.3 (2003): 277-283. EBSCOhost. Web. 22 September 2017.

[3] Lizhen Ji, Frans Oort, and Shing-Tung Yau. *The Legacy of Bernhard Riemann after One Hundred And Fifty Years* 1. International Press, Somerville, Massachusetts, 2016.

[4] Robert E. Greene and Kang-Tae Kim. *The Riemann Mapping Theorem from Riemann's Viewpoint*. Complex Analysis and its Synergies, 3.1 (2017): 1-11. MathSciNet. Web. 22 September 2017.