HOMEWORK 10

DUE NOV 20

We will write for two sets A, B, $A \approx B$ to mean that they have the same cardinality, which is the same as saying that they are bijective. We often say in this case A is *equivalent* to B.

For this homework, you may use facts from Calculus without proofs, but say what fact you are using.

We have defined a set A to be countable if there is an injection $A \to \mathbb{N}$ in class. We have also shown in class that if A is an infinite set and $a \in A$ a fixed element, then there is an injection $f : \mathbb{N} \to A$ with $f(1) = a$.

(1) Let A be an infinite set and let $B \subset A$ a finite set. Show that $A \approx (A - B)$. (Hint: $\mathbb{N} \approx \mathbb{N} - \{1\}$.)

(2) Let $A_n, n \in \mathbb{N}$ be a countable collection of sets (all contained in a universal set Ω).
   (a) Define $B_n = A_n - (A_1 \cup A_2 \cup \cdots \cup A_{n-1})$. Show that $\cup_{n=1}^{\infty} A_n = \cup_{n=1}^{\infty} B_n$ and $B_n \cap B_m = \emptyset$ if $n \neq m$.
   (b) If $A_n, n \in \mathbb{N}$ is a countable collection of countable sets, show that $\cup_{n \in \mathbb{N}} A_n = X$ is countable. (Hint: One way would be to use $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$).

(3) Let $\mathbb{Z}[X]$ denote the set of all polynomials in X with integer coefficients. Let $A_d \subset \mathbb{Z}[X]$ be the subset of all polynomials of degree at most d. (This means a polynomial $a_0 + a_1 X + \cdots + a_n X^n \in A_d, a_n \neq 0$ if and only if $n \leq d$.) Show that $A_d \approx \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$, $d+1$ copies. Deduce that $\mathbb{Z}[X]$ is countable.

(4) Show that $[a, b] \approx [c, d]$, closed intervals with $a < b, c < d$ of real numbers.

(5) Show that $[a, b] \approx (a, b)$, $a < b$.

(6) Show that $\mathbb{R} \approx (a, b)$, $a < b$. (Hint: Use log or exp judiciously.)

(7) Let $F(\mathbb{N}, \{0, 1\}) = X$ denote the set of all functions from $\mathbb{N}$ to the set $\{0, 1\}$ (set consisting of two elements $0, 1$). Define a map $\phi : X \to \mathbb{R}$ by $\phi(f) = \sum_{n=1}^{\infty} f(n) 10^{-n}$ for any $f \in X$. (We are assuming that all of you know that this is a convergent series.) Show that ϕ is an injection from X to $\mathbb{R}$. Deduce that $\mathbb{R}$ is not countable.