

HOMEWORK 11

DUE NOVEMBER 27

- (1) If $\{x_n\}$ is a CS, show that so is $\{|x_n|\}$.
- (2) Let $f : \mathbb{N} \rightarrow S$ be a sequence where S is any set. If $g : \mathbb{N} \rightarrow \mathbb{N}$ is an *increasing* function (i. e. $g(n+1) > g(n)$ for all n), we call the sequence $f \circ g : \mathbb{N} \rightarrow S$ a *subsequence* of f .
 - (a) Show that if f, g are as above, $g(n) \geq n$ for all $n \in \mathbb{N}$.
 - (b) Now, let $S = \mathbb{Q}$. Show that if f gives a CS, any subsequence is also a CS.
 - (c) Give an example (with $S = \mathbb{Q}$) to show that a sequence may not be a CS, but a subsequence can be a CS.
- (3) Let $\{x_n\}$ be a non-decreasing sequence (i. e. $x_{n+1} \geq x_n$ for all n). Assume the set $\{x_n | n \in \mathbb{N}\}$ is bounded above (i. e. there exists an M such that $x_n \leq M$ for all n). Show that $\{x_n\}$ is a CS.
- (4) Let $0 \leq b_n \leq a_n$ for all n . Let $x_n = \sum_{i=1}^n a_i$ and $y_n = \sum_{i=1}^n b_i$. Show that if $\{x_n\}$ is a CS, so is $\{y_n\}$.
- (5) Let $\{x_n\}$ be a CS with $x_n \geq r > 0$ for a fixed $r \in \mathbb{Q}$. Show that the sequence $\{\frac{1}{x_n}\}$ is also a CS. Give an example of a CS $\{x_n\}$ with $x_n > 0$ for all n , but $\{\frac{1}{x_n}\}$ is not a CS.
- (6) This is asking you to show that the harmonic series is not a CS.
 - (a) For any $n \in \mathbb{N}$, show that

$$\sum_{i=2^{n+1}}^{2^{n+1}} \frac{1}{i} \geq \frac{1}{2}.$$

- (b) Show that the sequence defined as $a_n = \sum_{i=1}^n \frac{1}{i}$ is not a CS. (Hint: You can use $\epsilon = \frac{1}{2}$).