HOMEWORK 12

(1) Show that the open interval \((a, b)\) \((a < b)\) is open and the closed interval \([a, b]\) is closed.

(2) Show that a subset \(X \subset \mathbb{R}\) is closed if and only if for any CS 
\( \{x_n\} \) with \(x_n \in X\), \(\lim x_n \in X\).

(3) Show that if \(U \subset \mathbb{R}\) is both open and closed, then \(U = \emptyset\) or
\(U = \mathbb{R}\). (This is usually referred to as \(\mathbb{R}\) is connected).

(4) Let \([a, b]\) \((a < b)\) be a closed interval and let \(U_n, n \in \mathbb{N}\) be open
sets of \(\mathbb{R}\) such that \([a, b] \subset \bigcup_{n \in \mathbb{N}} U_n\), called an open cover. Then
show that \([a, b] \subset \bigcup_{n=1}^{N} U_n\) for some \(N \in \mathbb{N}\). (This is usually
referred to as the closed and bounded interval being compact
and is a very important property).

(5) Let \(f : \mathbb{R} \rightarrow \mathbb{R}\) be a continuous function and assume that
\(f(x) \neq 0\) for all \(x \in \mathbb{R}\). Show that \(\frac{1}{f}\) is a continuous function
from \(\mathbb{R}\) to \(\mathbb{R}\).

(6) If \(f, g : \mathbb{R} \rightarrow \mathbb{R}\) are continuous functions, show that so is \(g \circ f\).

(7) Let \(f : \mathbb{R} \rightarrow \mathbb{R}\) be defined as \(f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots +
+ a_{n-1}x + a_1\), where \(n\) is an odd natural number and \(a_i \in \mathbb{R}\), some
fixed real numbers. (You are familiar with these, called poly-
nomial functions). Show that for \(x\) sufficiently large, \(f(x) > 0\)
and for all \(x\) sufficiently negative, \(f(x) < 0\). Conclude, using
intermediate value theorem that there is a real number \(a\) such
that \(f(a) = 0\).