HOMEWORK 12

(1) Show that the open interval (a, b) $(a < b)$ is open and the closed interval $[a, b]$ is closed.
(2) Show that a subset $X \subset \mathbb{R}$ is closed if and only if for any CS $\{x_n\}$ with $x_n \in X$, $\lim x_n \in X$.
(3) Show that if $U \subset \mathbb{R}$ is both open and closed, then $U = \emptyset$ or $U = \mathbb{R}$. (This is usually referred to as $\mathbb{R}$ is *connected*).
(4) Let $[a, b]$ $(a < b)$ be a closed interval and let $U_n, n \in \mathbb{N}$ be open sets of $\mathbb{R}$ such that $[a, b] \subset \cup_{n\in\mathbb{N}} U_n$, called an open cover. Then show that $[a, b] \subset \cup_{n=1}^{N} U_n$ for some $N \in \mathbb{N}$. (This is usually referred to as the closed and bounded interval being *compact* and is a very important property).
(5) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and assume that $f(x) \neq 0$ for all $x \in \mathbb{R}$. Show that $\frac{1}{f}$ is a continuous function from $\mathbb{R}$ to $\mathbb{R}$.
(6) If $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions, show that so is $g \circ f$.
(7) Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_1$, where n is an odd natural number and $a_i \in \mathbb{R}$, some fixed real numbers. (You are familiar with these, called polynomial functions). Show that for x sufficiently large, $f(x) > 0$ and for all x sufficiently negative, $f(x) < 0$. Conclude, using intermediate value theorem that there is a real number a such that $f(a) = 0$.