HOMEWORK 4, DUE OCT 2, 2017

(1) If $x \in \mathbb{R}$, $x \geq 0$ show that $(1+x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.

(2) Show that the sum of the first n odd natural numbers is n^2. That is, $1 + 3 + 5 + \cdots + (2n-3) + (2n-1) = n^2$.

(3) Show that if $n \geq 3$, $n \in \mathbb{N}$, $(1 + \frac{1}{n})^n < n$. (In fact a much stronger result is true, namely $(1 + \frac{1}{n})^n < 3$ for all $n \in \mathbb{N}$, but difficult to prove at this point.)

(4) Recall the definition of factorials. For any $n \in \mathbb{N}$, we define $n!$ (pronounced *n factorial*) to mean $1 \times 2 \times \cdots \times n$. By convention, we also define $0!$ to be 1. Show that for $n \geq 4$, $2^n < n!$. (Here is again a harder version, which you need not do, but at least mull over. If x is any positive real number, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $x^n \leq n!$).

(5) You may know the definition of the *greatest common divisor* (usually abbreviated by gcd) but I recall it here.

Given two integers a, b at least one of them non-zero, we say its gcd, written $\gcd(a,b)$ is a $d \in \mathbb{N}$, satisfying the following two properties:

(a) $d|a, d|b$.
(b) If $e \in \mathbb{N}$ with $e|a, e|b$, then $e|d$.

This problem shows the existence and uniqueness of $\gcd(a,b)$. I will outline the steps.

(a) Define $S = \{ma + nb \in \mathbb{N} | m, n \in \mathbb{Z}\}$. Show that $S \neq \emptyset$.
(b) By induction, we have a minimal element $d \in S$. We will show that it is $\gcd(a,b)$.
(c) Show that if $e|a, e|b$, then $e|d$.
(d) Use division algorithm to show that d divides both a and b. Thus, d is indeed the gcd.
(e) Show that if $d_1, d_2 \in \mathbb{N}$ satisfied both properties (a), (b) in the above definition, $d_1 = d_2$, proving the uniqueness of gcd.

(Thus, aposteriori, gcd could have been defined as the smallest natural number which can be written as $ma + nb, m, n \in \mathbb{Z}$ and this is how often it will be used.)