HOMEWORK 5, DUE OCT 9, 2017

(1) In this problem, we show the equivalence of the two induction procedures. So, let $N \in \mathbb{Z}$ be a fixed integer and let $A = \{n \in \mathbb{Z} | n \geq N\}$.
  (a) If S is a non-empty subset of A, it has a minimal element.
  (b) Let $P(n)$ for $n \in A$ be any open statement. Assume $P(N)$ is true and $P(n) \Rightarrow P(n+1)$ for any $n \in A$. Then, $P(n)$ is true for all $n \in A$.

  Show that the above two versions are equivalent by the following steps.
  (a) To show (a) implies (b), consider the set $S = \{n \in A | \neg P(n)\}$.
  (b) To show (b) implies (a), consider the open statement (the correct choice of an open statement is half the battle in induction proofs) $Q(n), n \in A$ to be,
  If $S \subset A$ with $m \in S$ with $m \leq n$, then S has a minimal element.
  Show that $Q(N)$ is true and $Q(n) \Rightarrow Q(n+1)$ for any $n \in A$ and deduce (a).

(2) Here is another variation of induction. So, let N, A be as above and let $P(n)$ be an open statement for $n \in A$. Assume $P(N)$ is true, $P(N+1)$ is true and $P(n) \Rightarrow P(n+2)$ is true for all $n \in A$. Show that $P(n)$ is true for all $n \in A$.

(3) Show that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ for any $n \in \mathbb{N}$.

(4) Let $n \in \mathbb{N}$ and let $s(n)$ be the sum of its digits (written the usual way). Show that $n \equiv s(n) \bmod 9$.

(5) Show that for any $n \in \mathbb{N}$, $4^n \not\equiv 2 \bmod 9$.