HOMEWORK 8

DO NOT SUBMIT

(1) We define relations on $\mathbb{Z}$ below, decide which ones are equivalence relations.
 (a) $a \sim b$ if $a + b$ is even.
 (b) $a \sim b$ if 3 divides $a + b$.
 (c) $a \sim b$ if $a \geq b$.
 (d) $a \sim b$ if $ab \geq 0$.

(2) If $f : A \to B$ is a function, we discussed an equivalence relation on A given by $a \sim a'$ if $f(a) = f(a')$. Such an equivalence relation is referred to as the equivalence relation *induced by* f.
 (a) Let $d \in \mathbb{N}$. Then we discussed in class the equivalence relation on $\mathbb{Z}$ defined as $a \sim b$ if $a \equiv b \pmod{d}$. Show that there is a *natural* function $f : \mathbb{Z} \to \{0, 1, \ldots, d - 1\}$ which is surjective and the equivalence relation is induced by f. (Here, the word *natural* is used to suggest that it should be more or less obvious, but even if it is not, once someone writes down the function, it should make you say 'Duh'.)
 (b) Fix a natural number d. Show that the relation on $\mathbb{R}$ given by $a \sim b$ if $a - b = kd$ for some $k \in \mathbb{Z}$ is an equivalence relation. Show that there is a natural surjective function $f : \mathbb{R} \to S^1$ (S^1 is the circle of unit radius, $x^2 + y^2 = 1$ in $\mathbb{R}^2$) and the relation is induced by f.
 (c) Define a relation on $\mathbb{R}$ by, $a \sim b$ if $a = \pm b$. Show that this is an equivalence relation and show that it is induced by a natural surjective function $f : \mathbb{R} \to \{x \in \mathbb{R} | x \geq 0\}$.

(3) Let $\sim$ be an equivalence relation on a set A and $\approx$ be an equivalence relation on a set B. Then define a relation $\equiv$ on $A \times B$ as follows. $(a, b) \equiv (a', b')$ if $a \sim a'$ and $b \approx b'$. Show that this is an equivalence relation.

(4) Let $\mathcal{C}^1(\mathbb{R})$ denote the set of all continuously differentiable functions on $\mathbb{R}$. Define an relation on this set by $f \sim g$ if $f' = g'$ (prime denotes differentiation). Is this an equivalence relation?