FINAL 310

TIME: TWO HOURS

You may use any result proved in class by quoting it precisely, unless the question asks you to prove something done in class, in which case you must prove it explicitly. You may also use elementary properties of numbers like the following without proof.

(1) Given any real number x, there exists an $N \in \mathbb{N}$ such that $N > x$.
(2) If $a \in \mathbb{R}$ with $0 \leq a < 1$ and $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $a^N < \epsilon$.

Do not answer more questions than asked for from each section. Do your scratch work separately and write as neatly as you can. Scratch out stuff you do not want me to look at.

Part A

Answer any two questions. Each is worth ten points

(1) Decide whether $P \Rightarrow (Q \wedge R)$ and $(P \wedge \neg Q) \Rightarrow R$ are logically equivalent by writing a truth table. (Your last two columns in the truth table should be the above two statements.)

(2) Write the converses and contrapositives of the following statements (either symbolically or in words).
   (a) $P \Rightarrow (Q \vee R)$.
   (b) $(\forall n \in \mathbb{Z})\ (n^2 + n + 1 = 0)$.
   (c) If $x, y \in \mathbb{Z}$ are even, then so is $x + y$.

(3) Decide, with proof, which of the following relations on $\mathbb{R}$ are equivalence relations.
   (a) $a \sim b$ if $a > b$.
   (b) $a \sim b$ if $a = b + n$ for some $n \in \mathbb{Z}$.

(4) Let A, B, C be subsets of Ω. Show that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Part B

Answer any two questions. Each is worth twenty points.

(5) If A, B are finite sets with $\#A = m, \#B = n$ show that $F(A, B)$, the set of all functions from A to B, is a finite set and calculate its cardinality.

(6) For this problem, we use the definition that a set S is infinite if there exists an injective map $f : \mathbb{N} \to S$. Show that if S is infinite and T is any non-empty set, $S \times T$ is infinite.

(7) If a, b are non-zero integers with $\gcd(a, b) = 1$, show that $\gcd(a+xb, b) = 1$ for any $x \in \mathbb{Z}$.

(8) Show that $\sum_{k=1}^{n}(2k - 1) = n^2$.

Part C

Answer any two questions. Each is worth twenty points.

(9) Let $\{x_n\}, \{y_n\}$ be two CS of real numbers with $\lim x_n = \lim y_n$. Show that $\{x_n^2\}, \{y_n^2\}$ are CS and $\lim x_n^2 = \lim y_n^2 = (\lim x_n)^2$.

(10) Let $x_n = \sum_{k=1}^{n} \frac{1}{k^3}$. Show that $\{x_n\}$ is an increasing sequence which is bounded above. Show that $\{x_n\}$ is a CS of real numbers.

(11) Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions and assume that $g(x) \neq 0$ for any $x \in \mathbb{R}$. Show that $F = f/g$ defined as $F(x) = f(x)/g(x)$ is also a continuous function.

(12) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and assume that for any $x, y \in \mathbb{R}$, $f(x+y) = f(x)+f(y)$. Then, show that $f(x) = f(1)x$ for any $x \in \mathbb{R}$.

Final 310

You may use any result proved in class by quoting it precisely, unless the question asks you to prove something done in class, in which case you must prove it explicitly. You may also use elementary properties of numbers like the following without proof.

(1) Given any real number x, there exists an $N \in \mathbb{N}$ such that $N > x$.
(2) If $a \in \mathbb{R}$ with $0 \leq a < 1$ and $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $a^N < \epsilon$.

Do not answer more questions than asked for from each section. Do your scratch work separately and write as neatly as you can. Scratch out stuff you do not want me to look at.

Part A

Answer any two questions. Each is worth ten points

(1) Decide whether $P \Rightarrow (Q \vee R)$ and $(P \wedge \neg Q) \Rightarrow R$ are logically equivalent by writing a truth table. (Your last two columns in the truth table should be the above two statements.)

(2) Write the converses and contrapositives of the following statements (either symbolically or in words).
  (a) $P \Rightarrow Q$.
  (b) If a non-zero integer a divides an integer b, then $2a$ divides $2b$.
  (c) If $x, y \in \mathbb{Z}$ are odd, then so is $x + y$.

(3) Decide, with proof, which of the following relations on $\mathbb{R}$ are equivalence relations.
  (a) $a \sim b$ if $a + b = 0$.
  (b) $a \sim b$ if $a = \alpha b$ for some $\alpha > 0$, $\alpha \in \mathbb{R}$.

(4) Let A, B, C be subsets of Ω and recall that $A - B = \{a \in A | a \notin B\}$ and $A + B = (A - B) \cup (B - A)$. Show that $(A + B) \cap C = (A \cap C) + (B \cap C)$.

Part B

Answer any two questions. Each is worth twenty points.

(5) If A, B are finite sets with $\#A = m, \#B = n$ and if $f : A \to B$ is an injective map, show that $m \leq n$.

(6) Given an $n \in \mathbb{N}$ and an infinite set S, show that there is an injective map $f : S \to S$ such that $S - f(S)$ is a finite set of cardinality n.

(7) If a, b are non-zero integers with $\gcd(a, b) = 1$, show that there exists $p, q \in \mathbb{Z}$ with $pa + qb = 1$ and $0 \leq p < b$.

(8) Show that $\sum_{k=1}^{n} \frac{1}{4k^2-1} = \frac{1}{2} - \frac{1}{2(2n+1)}$.

Part C

Answer any two questions. Each is worth twenty points.

(9) Let $\{x_n\}$ be a CS of rational numbers. Define a new sequence $\{y_n\}$ by $y_n = x_n x_{n+1}$. Show that $\{y_n\}$ is a CS. (Caution: No limits).

(10) Let $x_n = \sum_{k=1}^{n} \frac{1}{k^3}$. Show that $\{x_n\}$ is an increasing sequence which is bounded above. Show that $\{x_n\}$ is a CS of real numbers.

(11) Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions. Show that $F = fg$ defined as $F(x) = f(x)g(x)$ is also a continuous function.

(12) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and assume that for any $x, y \in \mathbb{R}$, $f(x+y) = f(x)+f(y)$. Then, show that $f(x) = f(1)x$ for any $x \in \mathbb{R}$.

Part D: Bonus question

(13) Let $\{x_n\}$ be a CS of real numbers with $\lim x_n = a$. Further assume that $|a| < 1$. Define a new sequence $\{y_n\}$ by $y_n = x_n^n$. Show that $\{y_n\}$ is a CS and $\lim y_n = 0$.

Final Examination, Math 310

Answer any six questions. All answers must be with proofs. If in doubt, ask. Please write legibly and write complete sentences. As always, use English, when in doubt about the correct use of symbols.

(1) Write the truth table for $A \wedge (A \Rightarrow B)$ and deduce that if this is true then B is true.

(2) Write the negations of the following, using only $\exists, \forall$ and not using any form of negations like $\sim, \notin$, words like *not* etc.
  (a) $\forall x \in \mathbb{R}, x^2 \geq 0$
  (b) $\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{R}$, $x > y$.

(3) Let $A, B \subset X$. Show that $A \subset B$ if and only if $B^c \subset A^c$.

(4) Let S be a non-empty set and let $\sim$ be an equivalence relation on it. Define a relation on the set $S \times T$, where T is any non-empty set as follows: (s, t) is related to (s', t') if $s \sim s'$. Show that this is an equivalence relation and describe the equivalence classes.

(5) Let $f : S \to T$ be a function, where S, T are non-empty sets. Let $F : S \to S \times T$ be defined as $F(s) = (s, f(s))$. Give conditions on S, T, f so that F is surjective.

(6) Let $f : \mathbb{R} \times \mathbb{N} \to \mathbb{R}$ be defined as, $f((a, b)) = a \cdot 10^b$. Is f injective? Is it surjective?

(7) Let M be a fixed natural number and let $S \subset \mathbb{Q}$ be a non-empty subset of positive rational numbers with the following property. Every $q \in S$ can be written as a/b with $a, b \in \mathbb{N}$ and $b \leq M$. Show that S has a minimal element, using the fact every non-empty subset of $\mathbb{N}$ has a minimal element.

(8) Prove the commutative law for addition in $\mathbb{N}$. You may only use the following: definition of addition, associativity of addition and the fact that $\mathbb{N} = \{1\} \cup \sigma(\mathbb{N})$.

(9) Let $\{x_n\}$ be a Cauchy sequence (say of rational numbers) such that there exists a number $\delta > 0$ and $\mid x_n \mid \geq \delta$ for all n. Show that $\{\frac{1}{x_n}\}$ is a Cauchy sequence.

(10) Show that $\{x_n\}$ where $x_n = \sum_{i=0}^{n} (-1)^i \frac{1}{2^i}$ is a Cauchy sequence. Show that $\lim x_n = \frac{2}{3}$.

Final, Math 310, April 30, 1999

Answer any eight questions. All answers must be with proofs. If in doubt, ask. Please write legibly and write complete sentences. As always, use English, when in doubt about the correct use of symbols.

(1) Write the truth tables for $A \Rightarrow B$ and $\sim B \Rightarrow \sim A$ and show that both have the same truth value.

(2) If A, B, C are sets such that $A \subset B$ and $B \subset C$, show that $A \subset C$.

(3) Assume that our univers is the set of integers. Rewrite the following using $\exists$ and $\forall$. (Do not use $\Rightarrow$).

  (a) If $m \in \mathbf{Z}$, we can find an integer n such that $m < n$.

  (b) If $m \in \mathbf{Z}$, then $m^2 \geq 0$.

  (c) Given $m \in \mathbf{Z}$, we can find an $n \in \mathbf{Z}$ such that $m < 2^n$.

(4) Let $S \subset \mathbf{Q}$, a non-empty subset of positive rational numbers. Assume that there exists a natural number N such that for any $s \in S$, $Ns \in \mathbf{N}$. Show that S has a minimal element.

(5) Let $S = \{1, 2, 3, 4\}$ and let $f : S \to S$ be defined as follows: $f(1) = 2, f(2) = 1, f(3) = 4, f(4) = 3$. Compute $f \circ f$.

(6) Let $\mathbf{R}^*$ be the set of non-zero real numbers. Define a relation on $\mathbf{R}^*$ as follows: $a \sim b$, if $ab \geq 0$. Is this an equivalence relation?

(7) Show that the sequence $\{x_n\}$, where $x_n = 1/(n^2+n)$ is Cauchy.

(8) Show that the sequence $\{x_n\}$ where $x_n = \sum_{i=0}^{n} 5^{-i}$ is Cauchy.

(9) Assume that given an integer N, there exists a natural number n such that $-n < N < n$. Use this to prove that given a rational number q, there exists a natural number n such that $-n < q < n$.

(10) Show that 7 doest not divide $n^2 - 3$ for any natural number n. (Hint: Do not use induction, but use remainders.)

(11) Show that $10^n \equiv 1 \bmod 9$ for all $n = 0, 1, 2, \ldots$. Thus compute the remainder of 654321 when divided by 9.

(12) (a) Prove that $\binom{n}{r} = \binom{n}{n-r}$.

  (b) Using the above, and binomial theorem, show that, if $n = 2m - 1$ an odd natural number, for numbers a, b

$$\begin{aligned} (a+b)^n &= \\ &= (a^n + b^n) + \binom{n}{1} ab(a^{n-2} + b^{n-2}) + \\ &\binom{n}{2} a^2 b^2 (a^{n-4} + b^{n-4}) + \cdots + \binom{n}{m} a^{m-1} b^{m-1} (a+b) \end{aligned}$$

(13) If a, b, c are integers so that $\gcd(a, b) = d$ and $\gcd(a, c) = 1$, then show that $\gcd(a, bc) = d$.

(14) Using Fermat's little theorem, compute the remainders of 3^{452} when divided by 7 and 2^{561} when divided by 11.

(15) Find (by experiment) a natural number x, $1 \leq x \leq 6$, such that, given any integer y, such that 7 does not divide y, there exists a natural number n with $x^n \equiv y \bmod 7$.

Final–Math 310

Answer any three questions from Part **A** *and any two from Part* **B**.
Try to write down all the steps.

A

(1) Show by induction that $2^{n-1} \leq n!$ for all $n \in \mathbb{N}$.
(2) Write down the contrapositive for $A \Rightarrow B$ and $A \wedge B$.
(3) If $f(X)$ is a polynomial with real coefficients and a is a real number, then show that $(X-a)^2$ divides $f(X)$ if and only if $f(a) = 0$ and $f'(a) = 0$, where f' denotes the derivative of f. (Hint: Easiest method is to use the division algorithm).
(4) If $z \in \mathbb{C}$ is a complex number with $\mid z \mid = 1$, then show that there exists a real number θ such that $z = \cos\theta + i\sin\theta$.
(5) Write down the three roots of the equation $x^3 = 1$.

B

(1) If p is a prime number which can be written as $a^3 + b^3$ with $a, b > 0$, integers, show that $p = 2$. (Hint: Does something divide $a^3 + b^3$?)
(2) Show that if $\{x_n\}$ is a Cauchy sequence and $f(X)$ is a polynomial with rational coefficients, then the sequence $\{y_n\}$ is also Cauchy, where $y_n = f(x_n)$. (Hint: Use the fact that addition and multiplication of Cauchy sequences are Cauchy and induction on the dgree of f.)
(3) Show that $\cos 72^o + i\sin 72^o$ is a root of the polynomial $x^4 + x^3 + x^2 + x + 1$. (Hint: Do not use the formula for quartic polynomials).

Math 310, Final, December 17, 2012, 10.30-12.30

All questions are worth ten points. Do any three each from Part I and Part II. Do any four from Part III. Please write clearly and in complete sentences. Scratch work has no place in a proof and you may loose points if you mix scratch work within a proof. Worse, I may not follow your proof.

1. Part I

(1) Write the truth table for $A \vee \neg B$ where A, B are mathematical statements.

(2) Write the converse and contrapositive of the statement,
  If a is even then $a+1$ is odd,
  without using 'not' or similar negations. Our universe is $\mathbb{Z}$ and you may use the fact that any integer is either even or odd, but not both.

(3) Decide (you do not have to give justifications) which of the following are true or false.
  (a) $(\forall a \in \mathbb{Z})(a^2 \geq 0)$.
  (b) $(\exists a \in \mathbb{R})(a^2 = -1)$.
  (c) $(\forall a \in \mathbb{R})(\exists b \in \mathbb{R})(b > a)$.

(4) Give a proof by contradiction to the statement: If $x, y \in \mathbb{R}$ and if $x^2 \neq y^4$ then $x \neq y^2$. (The proof should be short, less than five sentences, or you are probably doing something wrong).

2. Part II

(1) Prove that if A, B, C are sets, $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.

(2) Let $f : A \to B$ be an injective function from a set A to the set B. Define $g : \mathcal{P}(B) \to \mathcal{P}(A)$ where $\mathcal{P}$ as usual denotes power sets, by the following rule. If $X \in \mathcal{P}(B)$, then $g(X) = \{a \in A | f(a) \in X\}$. Prove that g is surjective.

(3) Let $f : A \to B$ be a function where A, B are sets. Prove that the relation on A given by, $a_1 \sim a_2$ for $a_1, a_2 \in A$, if $f(a_1) = f(a_2)$, is an equivalence relation.

(4) Prove that $\sum_{k=1}^{n}(2k-1) = n^2$ for any $n \in \mathbb{N}$.

3. Part III

(1) If $n, a \in \mathbb{N}$ with $2^n > a$, prove that $2^{n+1} > a + 1$, using only properties of natural numbers. (In particular, no subtractions, no zero).

(2) Prove the triangle inequality for integers.

(3) State the definition of $\gcd(a, b)$ for $a, b \in \mathbb{Z}$. Let $d > 0$ be an integer and let $a \in \mathbb{Z}$. If $a = qd+r$ with $q, r \in \mathbb{Z}$ and $0 \leq r < d$, prove that $\gcd(a, d) = \gcd(r, d)$.

(4) Prove that there is no rational number q with $q^3 = 2$. (You may use the fact that any rational number q can be written as a/b with $a, b \in \mathbb{Z}$ and $b \neq 0$ and $\gcd(a, b) = 1$).

(5) Prove that the sequence $\{x_n\}$, where $x_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}$ is a Cauchy sequence. (You may use the fact that give any (positive) number r, there exists an $N \in \mathbb{N}$ such that $N > r$).

(6) Prove that the sequence $\{x_n\}$ with $x_n = \sum_{k=0}^{n} q^n$ where $q \in \mathbb{Q}$ with $q > 1$ is not a Cauchy sequence. (You may use the fact that given any r there exists an $N \in \mathbb{N}$ such that $q^N > r$, where $q > 1$ as above).

Math 310, Final, December 19, 2011, 10.30-12.30

Please write legibly. If you do more problems than asked, indicate which I must grade. Do not do scratch work in the main body and cross it out so as not to confuse me.

PART I

All problems are worth 10 points each. Do any three.

(1) Write a truth table for the mathematical statement $(A \wedge B) \Rightarrow C$, where A, B, C are mathematical statements.

(2) Write the converse and contrapositive (in English, no symbols) of the following statement: If today is Monday, Lucy must be working.

(3) Write the negation of the statement (using the usual symbols), $(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(y > x)$.

(4) Rewrite the sentence $(\forall x \in \mathbb{Z})(x \text{ is even or } x \text{ is odd})$ without quantifiers.

(5) Let the universe of discourse be the set of integers. Consider the statements

$$P(x) : x \in \mathbb{Z} \text{ is a prime number,}$$

$$Q(x) : x \in \mathbb{Z} \text{ is even}$$

Decide with justification whether the statement $\exists x(P(x) \wedge Q(x))$ is true or false.

PART II

All problems are worth 15 points each. Do any two.

(6) Let A, B be subsets of a set X. If $A \subset B$ prove that $X - B \subset X - A$.

(7) If $f : A \to B$ and $g : B \to C$ are injective functions, prove that $g \circ f : A \to C$ is injective.

(8) Let $f : A \to B$ be a function and let $\Gamma \subset A \times B$ be its graph. Prove that A is bijective to Γ.

(9) Put a relation on $\mathbb{Z} \times \mathbb{Z}$ as follows. $(a, b) \sim (c, d)$ if $a + d = b + c$. Prove that this is an equivalence relation and the set of equivalence classes is naturally bijective to $\mathbb{Z}$.

Please turn over

PART III

All problems are worth 20 points each. Do any two.

(10) Prove by induction the formula, for $n \in \mathbb{N}$, $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

(11) Let $d, n \in \mathbb{N}$ with $d = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ and $n = p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k}$ where $p_1, p_2, \ldots, p_k$ are distinct primes and a_i, b_i are non-negative integers for all i, as assured by the fundamental theorem of arithmetic. Prove that d divides n if and only if $a_i \leq b_i$ for all i.

(12) Let $\Sigma_n = \{1, 2, \ldots, n\} \subset \mathbb{N}$ for any $n \in \mathbb{N}$. Prove that $\Sigma_m \times \Sigma_n$ for any $m, n \in \mathbb{N}$ is bijective to Σ_{mn}.

(13) Prove that the sequence $\{x_n\}$ where $x_n = \frac{1+n}{n}$ is a Cauchy sequence.

Final, Math 310, Fall 2009, 2hrs

The test has two parts. Answer any four questions from Part A and four from Part B. The questions in A are worth ten points each and the ones from B are worth fifteen each.

PART A

(1) Using only the closure, commutativity, associativity and distributivity properties of natural numbers, write a direct proof of the fact for any natural number a, $3a^2 + 3a + 7$ is odd, where as usual, a natural number n is even if $n = 2m$ for some natural number m and if n is not even, then $n + 1$ is even.

(2) Let P, Q, R be mathematical statements. Write a truth table for $(P \wedge Q) \Rightarrow R$.

(3) Write the following sentence and its negation in English without symbols. Determine with proof whether the statement is true or false.

$$(\forall x \in \mathbb{N})(\exists y \in \mathbb{Z})(y^2 < x).$$

(4) Write the converse and contrapositive to the following. If a, b, c are integers and $a^2 + b^2 = c^2$, then either a or b is even. Prove the statement or its converse or both, whichever is true.

(5) State any form of mathematical induction and prove the formula,

$$(1 - x)(1 + x + x^2 + \cdots + x^n) = 1 - x^{n+1}$$

for any real number x and n any natural number.

(6) If A, B, C are sets, show that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(7) Define injective and surjective functions from a set A to a set B. Show that if $f : A \to A$ is an injective map, where A is a finite set, then f is surjective.

PART B

(1) Assume for this problem, all human beings have one name. Define a relation by saying that two individulas are related if the first letter of their names are same. Show that this is an equivalence relation. How many distinct equivalence classes are there?

(2) Show that if S is a finite set with n elements, the the power set $\mathcal{P}(S)$ is finite and has 2^n elements.

(3) Define a Cauchy sequence of rational numbers and show that if $\{x_n\}$ is a Cauchy sequence, then there exists an $M \in \mathbb{Q}$ such that $|x_n| < M$ for all n.

(4) Show that the sequence $\{x_n\}$ defined as,

$$x_n = 1 + a + a^2 + \cdots + a^n,$$

where $a = 1/3$ is a Cauchy sequence. (You may use the fact that given an $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $a^N < \epsilon$.)

(5) Define the supremum of a non-empty set $S \subset \mathbb{R}$. Let

$$S = \{x \in \mathbb{R} \mid x > 0 \,\text{and}\, x^2 < 3\}.$$

Using theorems proved in class, show that S has a supremum and determine (a real number) this supremum.

(6) Define a continuous function $f : \mathbb{R} \to \mathbb{R}$ and show that if $\{x_n\}$ is a Cauchy sequence of real numbers and f is a continuous function, then $\lim f(x_n) = f(\lim x_n)$.

(7) Define the greatest common divisor. If p is a prime number and $a \in \mathbb{Z}$ with $a^n \equiv 1 \bmod p$, show that $a^d \equiv 1 \bmod p$ where $d = \gcd(n, p-1)$.