**Homework 8, Math 5031, Due Nov 9th**

1. Prove the following universal property of free modules. Let $F$ be an $A$-module and $S \subset F$. Then $F$ is free on $S$ if and only if for any $A$-module $M$ and a set map $f : S \to M$, there exists a unique $A$-module homomorphism $\phi : F \to M$ so that $\phi(s) = f(s)$ for all $s \in S$.

2. Show that $\mathbb{Q}$ is not a free $\mathbb{Z}$-module.

3. (a) Let $f : A \to B$ be a ring homomorphism of commutative rings with 1 (and $f(1) = 1$), so that $B$ is naturally an $A$-module. Let $M$ be any $A$-module. Consider the $A$-module $N = \mathrm{Hom}_A(B, N)$. Show that $N$ is naturally a $B$-module by the action, $(b\phi)(c) = \phi(bc)$ for any $b, c \in B$ and $\phi \in N$.

   (b) Take $A = k, B = k[x]/(x^n)$ and $M = k$ in the above problem. Let $\omega = \mathrm{Hom}_k(B, k)$ as in the above problem. Show that $\omega \cong B$ as $B$-modules.

4. Let $A$ be any commutative ring with 1 and let $S \subset A$ a multiplicatively closed subset.

   (a) Let $M$ be any $A$-module and let $i : M \to S^{-1}M$ the natural map. Show that for any $n \in S^{-1}M$, there exists an $m \in M$ and an $s \in S$ so that $i(m) = sn$.

   (b) Show that for any two modules $M, N$, there exists a natural $S^{-1}A$-module homomorphism,

   $$S^{-1}\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N).$$

   (c) (optional, just to think about) Is the above map injective? surjective? May be under some conditions?

5. (a) Construct an example of three abelian groups ($\mathbb{Z}$-modules) $M \overset{i}{\subset} N$ and a homomorphism $f : M \to P$ such that there is no homomorphism $g : N \to P$ so that $g \circ i = f$.

   (b) Show that given $M \overset{i}{\subset} N$ abelian groups with a homomorphism $f : M \to \mathbb{Q}$, there exists a homomorphism $g : N \to \mathbb{Q}$ so that $g \circ i = f$. (Hint: Use Zorn's lemma).