VEERING TRIANGULATIONS AND THE THURSTON NORM: HOMOLOGY TO ISOTOPY

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Abstract. We show that a veering triangulation $\tau$ specifies a face $\sigma$ of the Thurston norm ball of a closed 3-manifold, and computes the Thurston norm in the cone over $\sigma$. Further, we show that $\tau$ collates exactly the taut surfaces representing classes in the cone over $\sigma$ up to isotopy. The analysis includes nonlayered veering triangulations and nonfibered faces. We also prove an analogous theorem for manifolds with boundary that is integral to a theorem of Landry-Minsky-Taylor relating the Thurston norm to the veering polynomial, a new generalization of McMullen’s Teichmüller polynomial.

1. Introduction

Let $M$ be a closed, oriented, irreducible 3-manifold. A taut surface in $M$ is an embedded oriented surface which minimizes topological complexity in its homology class. Perhaps the simplest example is a fiber of a fibration of $M$ over the circle, if one exists. More generally, Thurston shows in [Thu86] that any compact leaf of a taut foliation of $M$ is taut. The converse is also true: in [Gab83], Gabai shows that any taut surface is a compact leaf of some taut foliation of $M$. The results herein show that veering triangulations are intimately related to taut surfaces and to the unit ball of the Thurston norm, an object of broad interest arising in areas from geometric group theory to Floer homology.

Before stating our results we give some broad-strokes definitions. A veering triangulation $\tau$ is a taut ideal triangulation of a torally bounded 3-manifold $\mathring{M}$ in the sense of Lackenby [Lac00] satisfying an extra condition on $\tau \cap \partial \mathring{M}$ (see Section 3.1). If $M$ is a Dehn filling of $\mathring{M}$ we define a natural element $e_\tau \in H_1(M)$ called the Euler class of $\tau$. The subset of $H_2(M)$ on which the Thurston norm $x$ agrees with $-e_\tau$ is either $\{0\}$ or the nonnegative cone over some face, which we name $\sigma_\tau$, of the Thurston norm unit ball $B_x(M)$. We denote this cone by $\text{cone}(\sigma_\tau)$. The 2-skeleton $\tau^{(2)}$ of $\tau$ is a cooriented branched surface in $M$ and an object we call a partial branched surface when viewed as a subset of $M$. In Section 2.3 we define these objects and say what it means for a partial branched surface to carry a surface. The set of homology classes of closed curves in $\mathring{M}$ positively transverse to $\tau^{(2)}$ generates a convex polyhedral cone $C_\tau \subset H_1(M)$ that we call the cone of homology directions of $\tau$. This defines a dual cone $C_\tau^\vee = \{ \alpha \in H_2(M) \mid \langle \alpha, \gamma \rangle \geq 0 \text{ for all } \gamma \in C_\tau \}$, where $\langle \cdot, \cdot \rangle$ denotes algebraic intersection.

The following two theorems summarize our main results.

**Theorem A.** Let $\tau$ be a veering triangulation of a compact 3-manifold $\mathring{M}$. If $M$ is obtained by Dehn filling each component of $\partial \mathring{M}$ along slopes with $\geq 3$ prongs then $M$ is irreducible and atoroidal. Let $\sigma_\tau$ be the face of the Thurston norm ball $B_x(M)$ determined by the Euler class $e_\tau$. Then $\text{cone}(\sigma_\tau) = C_\tau^\vee$, and the codimension of $\sigma_\tau$ in $\partial B_x(M)$ is equal to the dimension of the largest linear subspace contained in $C_\tau$.  

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Theorem B (Main theorem). Let \( M, \tau, \sigma \tau \) be as in Theorem A. A surface \( S \subset M \) is carried by \( \tau^{(2)} \) up to isotopy if and only if \( S \) is taut and \([S] \in \text{cone}(\sigma)\).

In the theorem statements we allow for the possibility that \( \sigma \tau \) is the empty face, which we assign dimension \(-1\). The term “prongs” describes a function for each component of \( \partial M \) from \{boundary slopes\} to \( \mathbb{Z}_{\geq 0} \), and is defined in Section 4.1. We restate these theorems at the end of Section 7 and show how they follow from intermediate results in the paper.

Theorem A demonstrates that if \( C \tau \subseteq H_1(M) \) then \( \tau \) is linked to the combinatorics of \( B_x(M) \) and to the Thurston norm itself. Theorem B says that \( \tau^{(2)} \) is a scaffolding off of which can be hung not only some taut representative of any homology class lying in \( \text{cone}(\sigma) \), but every taut representative of such a class up to isotopy. This is notable because while a fiber of a fibration \( M \to S^1 \) is the unique taut representative of its homology class up to isotopy ([Thu86, §3]), an integral class in \( H_2(M) \) is not necessarily represented by a unique taut surface up to isotopy and in general the collection of taut surfaces representing a single homology class is not well understood. In fact it is a general phenomenon that one might wish to prove some statement about all taut representatives of a homology class and succeed only in proving the statement holds for a single taut representative of the homology class. We describe two examples of this in order to motivate the promotion of homology to isotopy in general.

Example 1. The Fully Marked Surface Theorem [GY20, Theorem 1.1] of David Gabai and Mehdi Yazdi says that if \( F \) is a taut foliation of \( M \) and \( S \) is fully marked (meaning that its Euler characteristic is given by pairing with the Euler class of \( F \)), then the tangent bundle of \( F \) is homotopic to the tangent bundle of a new taut foliation \( F' \) such that \( S \) is homologous to a compact leaf \( S' \) of \( F' \). Gabai and Yazdi conjecture [GY20, Conjecture 1.5] that it is not possible to replace “homologous” with “isotopic.”

Example 2. Lee Mosher proves the Transverse Surface Theorem [Mos92, Theorem 1.3.1] over the course of [Mos89, Mos90, Mos91, Mos92], which says that if \( \phi \) is a pseudo-Anosov flow on \( M \), then any integral homology class in \( H_2(M) \) which pairs nonnegatively with the homology class of each closed orbit of \( \phi \) is represented by a taut surface which is almost transverse (see [Mos92, §1]) to \( \phi \). It is not known if any taut surface representing such a homology class is almost transverse to \( \varphi \) up to isotopy, although Mosher was able to achieve a partial result in this direction [Mos92, §3].

1.1. Related work on flows. Unwritten work of Ian Agol and François Guéritaud shows that a pseudo-Anosov flow with no perfect fits (see e.g. [Fen12, Definition 2.2]) gives rise to a veering triangulation together with a Dehn filling, where the cores of the filling tori correspond to singular orbits and the combinatorial notion of prongs from Theorem A corresponds to the number of prongs of a singular orbit. Work in progress of Saul Schleimer and Henry Segerman ([SS19] and more to come) aims to show the reverse, i.e. that the pseudo-Anosov flow can be reconstructed from the triangulation and the filling. Mosher has shown [Mos92, Flows Represent Faces] that under certain conditions (in particular the case of no perfect fits), a pseudo-Anosov flow on \( M \) represents a face of \( B_x(M) \) in the sense that its cone of homology directions [Mos92, §1.2] is dual to the cone on which its negative Euler class agrees with the Thurston norm. His Transverse Surface Theorem (see Example 2 above) then implies every integral class in this cone is represented by a surface almost transverse to the flow.

Here is how our results fit into this picture:

- If one takes for granted the results in progress above, the statement of Theorem A is unsurprising in light of Mosher’s results. Its interest lies in the fact that the statement...
and its proof are entirely independent of pseudo-Anosov flows and the work of Mosher, Agol-Guéritaud, and Schleimer-Segerman. Moreover the proof introduces the ideas of flattening and flat isotopy (see Section 4). These ideas are later applied to prove Theorem 8.1, which is key to a main result in \[LMT20\] relating the Thurston norm to the veering polynomial, an object generalizing McMullen’s Teichmüller polynomial [McM00].

- Being carried by \(\tau^{(2)}\) is a combinatorial version of almost transversality. However, Theorem B is a much stronger result than could be obtained from simply viewing the above work through a combinatorial lens, even if one knew independently that being almost transverse to the flow implied being carried by \(\tau^{(2)}\) (this last fact is implied by our results). The key point is the distinction between homology and isotopy.

1.2. Veering triangulations. Veering triangulations, which can be thought of as combinatorial versions of both foliations and flows, have enjoyed much recent interest. As these triangulations play a prominent role in the paper and are still perhaps esoteric to many readers, we survey some of the literature to date. Veering triangulations are introduced by Agol in [Ago10] with the goal of analyzing the mapping tori of pseudo-Anosov homeomorphisms with small dilatation. The triangulations he studies and constructs are all layered, meaning they are built from stacking tetrahedra on a surface and taking a quotient by some homeomorphism of the surface. In the literature on veering triangulations, few papers ([HRST11, FG13, SS19, LMT20] and this one) include analyses of the nonlayered case. Layered veering triangulations have been used to study Cannon-Thurston maps [Gué16], the curve and arc complexes of surfaces [MT17, Str18], and pseudo-Anosov flows on fibered hyperbolic 3-manifolds [Lan18, Lan19]. There is another branch of study concerning the question of when a veering triangulation admits a geometric structure, or more generally a strict angle structure [HRST11, FG13, HIS16, FTW18]. More recently there is the work of Agol-Guéritaud and Schleimer-Segerman described in Section 1.1.

1.3. Applicability, methods, and outline. The work of Agol-Guéritaud described in Section 1.1 shows that our results apply in any closed 3-manifold supporting a pseudo-Anosov flow with no perfect fits. By results of Calegari, any atoroidal 3-manifold admitting an \(\mathbb{R}\)-covered taut foliation [Cal00, Corollary 5.3.16] or more generally a taut foliation with 1-sided branching [Cal01, Corollary 4.2.9] admits a pseudo-Anosov flow transverse to the foliation. By results of Fenley [Fen12, Theorem G and Theorem H] these flows have no perfect fits. Hence our theorem can be applied in any atoroidal manifold admitting an \(\mathbb{R}\)-covered foliation or more generally a taut foliation with one-sided branching. The result also applies to fibered hyperbolic 3-manifolds using Agol’s original construction; another way to see this by noting that the suspension flow of a pseudo-Anosov map is a pseudo-Anosov flow with no perfect fits.

It is natural to ask to what extent Theorems A and B can be used to compute \(x\) and classify faces of \(B_x(M)\). More specifically we can ask whether, given a taut surface in \(M\), one can find a veering triangulation of a link exterior \(\hat{M}\) in \(M\) such that the filling \(\hat{M} \to M\) satisfies the \(\geq 3\) prongs condition. This remains unresolved for the time being.

We now describe the proofs of Theorems A and B. The containment \(C_\tau^x \subset \text{cone}(\sigma_\tau)\) in Theorem A follows from Theorem 5.5, which is a combinatorial version of Mosher’s Transverse Surface Theorem. The proof of Theorem 5.5 relies on a result from [LMT20] which we modify for our setting using techniques from our earlier paper [Lan18]. The proof of the reverse containment is considerably more involved. In the course of the proof we take a taut surface
2.1. **Euler characteristic, train tracks, index.** Let $S$ be a compact surface, possibly with boundary, and let $t$ be a train track on $S$. By **train track**, we mean a 1-complex properly embedded in $S$ with a tangent space at each point varying continuously. We require that $t$ intersect the boundary of $S$ transversely. Following [PH92] we call the points of $t \cap \partial S$ **stops**. The 1-cells of $t$ are called **branches** and the nodes of valence $> 1$ are called **switches**. We make no assumptions about the types of complementary regions of $t$ or the valence of nodes in this notion of train track; however, we say a train track is **generic** if its switches are all trivalent. If a train track $t$ is generic, there is a vector field defined on the set of switches of $t$ defined up to scaling by the property that at each switch $p$ the vector field points away from the two-sheeted side and toward the one-sheeted side of $p$. This is called the **maw vector field**. Often, to describe the direction of the maw vector field at a certain switch, we will say that the **switch** points in that direction. If $b$ is a branch of $t$ such that neither endpoint of $b$ is a stop and the switches at both endpoints point into $b$, we say $b$ is a **large branch** of $t$.

A **patch of** $(S,t)$, or simply a **patch of** $t$ if the surface in question is clear from context, is the closure of a component of $S - t$ with respect to a path metric. A patch of $t$ is topologically a surface with boundary, but a patch has additional data: the boundary of a patch may have cusps, which we call the **switches** of the patch, and there may also be **corners**. The switches and corners of a patch are the places where the patch’s boundary is
not $C^1$ and they correspond respectively to points where either $t$ has a switch or meets the boundary of $S$. (We use the word switch instead of cusp because we wish to reserve cusp for a different usage later in the paper).

Define the index of a patch $p$ to be twice its Euler characteristic, minus its number of switches, minus half its number of corners:

$$\text{index}(p) = 2\chi(p) - \#\{\text{switches of } p\} - \frac{\#\{\text{corners of } p\}}{2}.$$ 

For example, the index of a disk with 2 boundary cusps is $2(1) - 2 = 0$. The Euler characteristic of $S$ can be obtained from the indices of its patches:

$$2\chi(S) = \sum \text{index}(p),$$

where the sum is taken over the patches of $t$.

A patch which is topologically a disk and has $n$ switches and no corners will be called an $n$-gon patch or simply an $n$-gon. In particular we will refer to nullgons, monogons, and bigons in our arguments.

A patch which is topologically an annulus with no switches or corners in its boundary is called an annulus. To reduce confusion, if we wish to talk about a patch which is topologically an annulus but may have switches in its boundary we will refer to it as a topological annulus.

2.2. Branched surfaces. A branched surface is a 2-complex with a continuously varying tangent plane at every point, locally modeled on the quotient of a stack of disks by identifying closed half-disks (see [Oer86] for more detail). Branched surfaces should be thought of as akin to train tracks: just as a train track embedded in a surface can carry curves and laminations with 1-dimensional leaves, a branched surface embedded in a 3-manifold can carry surfaces and laminations with 2-dimensional leaves.

We now describe what it means for a branched surface to carry a surface. A regular neighborhood $N(B)$ of a branched surface $B$ has a foliation $\mathcal{F}$ whose leaves are line segments intersecting $B$ transversely. If $\mathcal{F}$ can be oriented, we say $B$ is a cooriented branched surface. If $S$ is an oriented surface embedded in the oriented 3-manifold $M$, there is a natural coorientation on $S$ determined by requiring that a coorientation vector at a point $p$ in $S$, when appended to a positive basis for the tangent space $T_pS$, gives a positive basis for
Figure 2. The two local models for a branched surface with generic branch locus.

$T_p M$. Suppose $B$ is a cooriented branched surface. If $S$ is embedded in $N(B)$ transverse to $\mathcal{F}$ so that the orientation of $\mathcal{F}$ agrees with the coorientation of $S$, we say $S$ is carried by $B$.

The non-manifold points of a branched surface $B$ form its branch locus, denoted $\text{brloc}(B)$. The connected components of $B - \text{brloc}(B)$ are called sectors of $B$. A branch curve of a branched surface $B$ is the image of an immersion $S^1 \to \text{brloc}(B)$. A branch segment is the image of an immersion $[0, 1] \to \text{brloc}(B)$.

A branched surface with generic branch locus is a branched surface $B$ locally modeled on one of the two spaces shown in Figure 2, which should be distinguished since our branched surfaces will always be embedded in orientable 3-manifolds. The non-manifold points of $\text{brloc}(B)$ for a branched surface with generic branch locus are called triple points.

Note that if $B$ is a branched surface in a 3-manifold $M$, and $S \subset M$ is a properly embedded surface in general position relative to $B$, then $B \cap S$ is a train track in $S$ and it makes sense to speak of the patches of $B \cap S$. If $\text{brloc}(B)$ is generic, then $B \cap S$ is a generic train track.

We conclude this section by proving a basic lemma about incompressible surfaces in relation to branched surfaces in irreducible 3-manifolds. If $S$ is a surface embedded in $M$ in general position relative to a branched surface $B$, we say $S$ has simple patches if each patch of $S \times B$ is $\pi_1$-injects into its component of $M - B$.

**Lemma 2.2** (simple patches). Let $B$ be a branched surface in an irreducible 3-manifold $M$. Let $S$ be an incompressible surface embedded in $M$. Then $S$ can be isotoped so that $S \cap B$ has simple patches

**Proof.** Suppose $P$ is a patch of $S \cap B$ in a component $T$ of $M - B$ and suppose that $P \to T$ is not $\pi_1$-injective. By the loop theorem there is some curve $\gamma \subset P$ which is essential in $P$ and bounds an embedded disk in $T$. If $\gamma$ does not bound a disk in $S$ then $D$ is a compression disk for $S$, a contradiction. Otherwise $\gamma$ bounds a disk $D'$ in $S$. Since $M$ is irreducible, $D \cup D'$ bounds a ball and by an innermost disk argument there is an isotopy of $S$ moving $D'$ into $T$. The effect on $S \cap B$ is to delete $D' \cap B$, strictly reducing the number of patches of $S \cap B$. After finitely many of these operations our surface will be of the desired form. $\square$

2.3. Partial branched surfaces. We will also need to consider objects called partial branched surfaces, which are slightly more general than branched surfaces and to our knowledge have not been studied before. These are branched surfaces properly embedded in a submanifold of a 3-manifold.

**Definition 2.3.** Let $M$ be a compact 3-manifold.
Figure 3. A taut tetrahedron with edge angles and face coorientations indicated. The tips colored green are upward flat triangles, and the cyan tips are downward flat triangles.

a. A partial branched surface in $M$ is a 2-complex $B$ in $M$ such that there exists a union of solid tori $U \subset M$ such that $B$ is a properly embedded branched surface in $M - U$.

b. An embedded surface $S \subset M$ is carried by $B$ if $S$ can be isotoped so that $S - U$ is carried by $B - U$ in $M - U$ and $S \cap U$ is a union of disks and annuli which $\pi_1$-inject into their respective components of $U$.

2.4. The Thurston norm, its dual, and taut surfaces. For a reference elaborating on the following see [Thu86]. Let $M$ be a closed, oriented, irreducible, atoroidal 3-manifold. If $\alpha$ is an integral point in $H_2 \bigotimes \mathbb{Q}$, define

$$x(\alpha) = \min\{-\chi(S - \text{sphere components}) \mid S \text{ embedded, oriented, and } [S] = \alpha\}.$$  

Thurston shows that $x$ extends to a norm on the vector space $H_2(M)$, now known as the Thurston norm, and that its unit ball $B_x(M)$ is a finite-sided rational polyhedron.

The dual Thurston norm is the norm $x^*$ on $H_1(M)$ defined by

$$x^*(\gamma) = \sup\{\langle\alpha, \gamma\rangle \mid x(\alpha) \leq 1\},$$

where $\langle\cdot, \cdot\rangle$ is algebraic intersection. We let $B_{x^*}(M) = \{\gamma \in H_1(M) \mid x^*(\gamma) = 1\}$. Thurston shows that $B_{x^*}(M)$ is a polyhedron with integral vertices.

We say an embedded surface $S \subset M$ is taut if $-\chi(S) = x([S])$ and no component of $S$ is nullhomologous. Any taut surface is incompressible, since a compression surgery increases Euler characteristic.

More generally, Thurston shows that if $M$ admits surfaces of nonnegative Euler characteristic representing nonzero homology classes then $x$ gives a pseudonorm on $H_2(M)$. However, we will not need to consider this case.

3. Veering triangulations: background and preliminaries

3.1. Definition of veering triangulation. A taut tetrahedron is an ideal tetrahedron with the following extra structure: two faces are cooriented outward, two are cooriented inward, and the edges are labeled 0 and $\pi$ according to whether face coorientations disagree or agree along that edge, respectively. By pinching along the edges labeled 0 and smoothing along the edges labeled $\pi$, we always think of the 0, $\pi$ labels as denoting the interior angle of the taut tetrahedron at that edge. This gives a taut tetrahedron a smooth structure in the sense that there is a well-defined tangent plane at every point in the boundary. See Figure 3.
A face of a taut tetrahedron $\mathcal{O}$ is called a **top (bottom) face** if it is cooriented outward (inward). The union of the two top (bottom) faces of $\mathcal{O}$ is called the **top (bottom) of** $\mathcal{O}$. The edge along which the two top (bottom) faces of $\mathcal{O}$ meet is called the **top (bottom) edge** of $\mathcal{O}$.

A **taut ideal triangulation** is an ideal triangulation $\tau$ of a compact 3-manifold $N$ by taut ideal tetrahedra such that for each edge $e$, the sum of interior angles around $e$ is $2\pi$ (equivalently, $e$ is the image of exactly two edges labeled $\pi$ under the quotient map $\{\text{tetrahedra}\} \to N$) and such that bottom faces are only identified with top faces and vice versa. This forces $N$ to be orientable, and a choice of orientation for $N$ gives an orientation on the 2-skeleton $\partial_2 \tau$ by requiring that the direct sum of the orientation on each tangent plane with the coorientation be positively oriented. This gives a notion of clockwise and counterclockwise on each 2-cell of $\tau$.

Let $\tau$ be a taut ideal triangulation. By removing small neighborhoods of the ideal vertices, we will always think of $\tau$ as a decomposition of a compact manifold $\tilde{M}$ into **truncated** taut tetrahedra. A truncated taut tetrahedron $\mathcal{O}$ has 8 sides, 4 of which are hexagons and 4 of which are triangles which we call the **tips** of $\mathcal{O}$. Each tip has the smooth structure of a bigon. Instead of repeating “truncated taut tetrahedron,” we will use the term $\tau$-**tetrahedron**. Our convention will be that the terms $\tau$-**face** and $\tau$-**edge** refer to “honest” faces and edges of $\tau$, i.e., faces and edges intersecting $\text{int}(\tilde{M})$. If $\triangle$ is a $\tau$-face, each edge of $\triangle$ which is not a $\tau$-edge is called a **tip** of $\triangle$ (note that $\triangle$ has 3 tips).

The 2-skeleton $\partial_2 \tau$ of $\tau$ can naturally be viewed as a cooriented branched surface. See Figure 4. The branch locus of $\partial_2 \tau$ is equal to the collection of $\tau$-edges. Note that $\tau$ induces a cooriented train track

$$\partial \tau := \tau \cap \partial \tilde{M}$$

on $\partial \tilde{M}$. The patches of $\partial \tau$ are all tips of $\tau$-tetrahedra, so for Euler characteristic reasons $\partial \tilde{M}$ must be a union of tori. Let $U := \text{int}(\tilde{M})$, so $U$ is a union of closed solid tori.

A **flat triangle** $t$ is a bigon of a cooriented train track $T$ with three branches of $T$ in its boundary. An example is a tip of a $\tau$-tetrahedron with respect to $\partial \tau$. The flat triangle $t$ has three vertices, two of which are switches of $t$; the internal angle is $0$ at these vertices, which we call **cuspidal**. We will call the third vertex **noncuspidal**; the internal angle at this vertex is $\pi$. An edge of a flat triangle is a **0-0 edge** if it connects the two cuspidal vertices and otherwise is a **0-$\pi$ edge**. There are two types of flat triangles, distinguished by the coorientation at the noncuspidal vertex: A flat triangle $t$ is called **upward** (**downward**) if the coorientation at the noncuspidal vertex points out of (into) $t$. See Figure 5.
An upward (downward) ladder is an annulus $A$ together with a cooriented train track $t$ containing $\partial A$ such that all patches of $t$ are upward (downward) flat triangles and for each branch $b$ of $t$, either $b \subseteq \partial A$ or $(b - \{\text{endpoints of } b\}) \subseteq \text{int } A$ and both endpoints of $b$ lie on different components of $\partial A$. A rung is a branch of $t$ with endpoints on different components of $\partial A$.

A cooriented train track $t$ on a torus $T$ is pseudohyperbolic if $T$ is a union of ladders with disjoint interiors such that each upward (downward) ladder is disjoint from all other upward (downward) ladders.

**Definition 3.1.** A taut ideal triangulation $\tau$ of $\tilde{M}$ is veering if it induces a pseudohyperbolic train track on each boundary component of $\tilde{M}$.

This definition is equivalent to Agol’s original definition in [Ago10] and the definition involving an edge bicoloring introduced by Hodgson-Rubinstein-Segerman-Tillmann in [HRST11] which is now common. We choose this definition since the boundary train tracks of veering triangulations are integral to our arguments. To see that the definitions are equivalent, one notes that Agol’s veering condition on edges corresponds to a certain condition on $\partial \tau(2)$ (see [Lan18, Lemma 2.6]), which implies the ladder behavior described here.

### 3.2. Veering combinatorics: ladderpoles, left/right veer, fans

Many of the terms defined in this subsection were coined in [FG13].

Let $T$ be a torus with a pseudohyperbolic train track. The boundary components of the ladders in $T$ are called ladderpoles. Together, the ladderpoles form an even-sized collection of parallel curves and determine a slope on $T$ called the ladderpole slope.

Let $\tau$ be a veering triangulation and $T$ be a component of $\partial \tilde{M}$. We define a notion of left and right on $T$ as follows. Let $L$ be a downward ladder; orient the core curve of $L$ so that it intersects each rung positively. If we look at $L$ from inside $\tilde{M}$ so that our heads point in...
Figure 7. In this picture we have drawn all the $\tau$-tetrahedra incident to a bold $\tau$-edge in the universal cover $\hat{M^\circ}$. The $\tau$-tetrahedra which are not part of the two fans of the bold edge are outlined in gray. The bold $\tau$-edge is incident to a long fan and a short fan. We have colored the flat triangles incident to a boundary point of the dark $\tau$-edge to indicate their hinge- or non-hingeness: the yellow flat triangles are hinge and the pink flat triangles are non-hinge. The reader may wish to check that the bold edge is left veering.

The direction of the core curve, the ladderpole of $L$ to our left (right) is called the left (right) ladderpole of $L$.

A left (right) ladderpole of $\tau$ is a ladderpole which is the left (right) ladderpole of a downward ladder. One can check that for every $\tau$-edge $e$, either both endpoints of $e$ lie in left ladderpoles or both lie in right ladderpoles. If both endpoints of $e$ lie in left (right) ladderpoles, then $e$ is left (right) veering. The quality of being left or right veering is called the veer of an edge. A $\tau$-tetrahedron is called hinge if its top and bottom edges have opposite veer, and non-hinge otherwise. We will also say that a tip of a $\tau$-tetrahedron $\otimes$ is hinge or non-hinge if $\otimes$ is hinge or non-hinge, respectively.

As with our definition of veering triangulation, this definition of veer agrees with others in the literature, and we have chosen to phrase it in this way in order to highlight what is most relevant to our arguments.

Let $\hat{M^\circ}$ be the universal cover of $\hat{M}$ and let $\hat{\tau}$ be the lift of $\tau$ to $\hat{M}$. We will refer to $\hat{\tau}$-tetrahedra, $\hat{\tau}$-faces, and $\hat{\tau}$-edges. It follows from [SS20, Theorems 3.2 and 5.1] that no two $\hat{\tau}$-edges or $\hat{\tau}$-faces of a single $\hat{\tau}$-tetrahedron are identified, which makes it easier to define certain objects in $\hat{M}$ by first describing an object in $\hat{M^\circ}$ and then projecting to $\hat{M}$.

Let $e$ be a $\tau$-edge. If we circle around $e$ and read off the interior angles at $e$ of $\tau$-tetrahedra incident to $e$, there are two $\pi$ angles and some number of $0$ angles. It is a consequence of our veering definition that the two $\pi$-angles are not circularly adjacent. Hence the $\hat{\tau}$-tetrahedra whose interior angles at $e$ are $0$ are split into two nonempty sets, each of which is called a fan of $e$. A fan is short if it consists of one $\hat{\tau}$-tetrahedron and long otherwise.

Let $v$ be a switch of $\hat{\tau}$ corresponding to a $\tau$-edge $e$. A fan of $v$ is the union of all upward (or downward) flat triangles for which $v$ is a cuspidal vertex. This corresponds exactly to the intersection of one of the fans of $e$ with the component of $\partial\hat{M^\circ}$ containing $v$. A fan of $v$ is short if it consists of only one flat triangle and long otherwise. In $\hat{M}$, we define short and
long fans of \( \tau \)-tetrahedra and flat triangles to be the images of the corresponding objects in \( \tilde{M} \) under the covering projection.

Note that the coorientation on \( \partial \tau^{(2)} \) allows us to speak of the \textbf{topmost} or \textbf{bottommost} flat triangle in a particular fan. We now record some facts about fans and hingeness from [FG13] in the following lemma.

**Lemma 3.2 (FG13).** A non-hinge flat triangle is incident to a short fan at one of its cuspidal vertices. At the other cuspidal vertex, it is part of a long fan for which it is neither the top nor bottom flat triangle.

A hinge flat triangle is topmost in a fan corresponding to one of its cuspidal vertices, and bottommost in a fan corresponding to its other cuspidal vertex.

**Remark 3.3.** In [SS19], \( B^s \) and \( B^u \) are called the \textit{upper and lower branched surfaces in dual position} and are colored green and purple respectively. We have chosen to color them red and blue in analogy with the stable and unstable singular foliations of a pseudo-Anosov flow, which are usually colored red and blue.
The dual graph of $\tau$, denoted $\Gamma$, is the 1-skeleton of the complex dual to $\tau$. There is a natural orientation on $\Gamma$, defined by the property that the orientation on each edge agrees with the coorientation of the corresponding 2-cell of $\tau$. Let $\tilde{\Gamma}$ denote the lift of $\Gamma$ to $\tilde{M}$. Since $B^s$ and $B^u$ are two different smoothings of the same 2-complex, they are isotopic in $\tilde{M}$. Their branch loci can both be naturally identified with $\Gamma$ such that triple points are identified with $\Gamma$-vertices. Thus we will always think of the branch curves and segments of $B^s$ and $B^u$ as being oriented compatibly with $\Gamma$.

We now need to discuss how $B_s$ and $B_u$ interact with the veers of $\tau$-edges. As before, it is convenient to work in $\tilde{M}$. Let $\triangle$ be a $\bar{\tau}$-face. Then $\triangle^s := \triangle \cap \tilde{B}^s$ is a train track with a single generic switch, as is $\triangle^u := \triangle \cap \tilde{B}^u$. These train track switches recover the veer of $\bar{\tau}$-edges and behave in a controlled way with respect to $\partial \bar{\tau}^{(2)}$, as we describe in the next two lemmas. Each lemma is most easily understood via the included pictures, but we also provide formal statements.

**Lemma 3.4** (train tracks from veering data). Let $\triangle$ be a $\bar{\tau}$-face. Then $\triangle^s = \triangle \cap \tilde{B}^s$ and $\triangle^u = \triangle \cap \tilde{B}^u$ can be recovered from the veer of the $\bar{\tau}$-edges of $\triangle$ as in the following picture, where the coorientation points out of the page and we have drawn $\triangle^s$ in red and $\triangle^u$ in blue.

In words:

- If $\triangle$ has two left veering $\bar{\tau}$-edges $e_1, e_2$ labeled so that $e_2$ follows $e_1$ in the counterclockwise order of the $\bar{\tau}$-edges of $\triangle$, then the switch of $\triangle^s$ points toward $e_1$ and the switch of $\triangle^u$ points toward $e_2$.
- If $\triangle$ has two right veering $\bar{\tau}$-edges $e_1, e_2$ labeled so that $e_2$ follows $e_1$ in the counterclockwise order, then the switch of $\triangle^u$ points toward $e_1$ and the switch of $\triangle^s$ points toward $e_2$.

**Proof.** The reader can check this in order to get comfortable with the combinatorics of veering triangulations.

**Remark 3.5.** We emphasize here that while we have specified the position of $B^s$ and $B^u$ relative to $\tau^{(2)}$, we have not specified the position of $B^s$ and $B^u$ relative to each other, and we will not do so. The salient feature of the train tracks in the lemma statement is the configuration of each train track relative to the right and left veering edges, not relative to the other train track.

Now suppose that $\triangle$ is a $\bar{\tau}$-face and suppose its tips are labeled $t_1, t_2, t_3$. Each $t_i$ determines a unique patch $t_i^s$ of $\triangle \cap \tilde{B}^s$ and $t_i^u$ of $\triangle \cap \tilde{B}^u$. Note that for each $i$, at most one of $t_i^s, t_i^u$ has a switch.

**Lemma 3.6.** With notation as above, $t_i$ is a ladderpole branch of $\partial \bar{\tau}^{(2)}$ if and only if $t_i^s$ and $t_i^u$ have no switches, and a rung of $\partial \bar{\tau}^{(2)}$ crossing an upward (downward) ladder if and only if the patch $t_i^s$ ($t_i^u$) has a switch. (See Figure 9.)

**Proof.** We recall that both ends of a left (right) veering edge meet $\partial \tilde{M}$ in the left (right) ladderpole of a downward ladder. The lemma is now a consequence of Lemma 3.4.
3.4. Branch curves, stable loops, unstable loops. Following [SS20] we define a normal curve for $\tau$ to be a smoothly immersed loop $\gamma: S^1 \to \tau(2)$ which is transverse to $\tau$-edges and normal in each $\tau$-face, i.e. if $\hat{\gamma}$ is a lift of $\gamma$ to $\hat{\mathcal{M}}^\circ$ then for each $\hat{\tau}$-face $\Delta$ intersected by $\hat{\gamma}$, each component of $\hat{\gamma}^{-1}(\Delta)$ has its endpoints on different $\hat{\tau}$-edges (in fact, since Schleimer and Segerman prove normal curves are homotopically nontrivial, there will be only one component of $\hat{\gamma}^{-1}(\Delta)$).

Note that for any $\hat{\tau}$-edge $e$ and for each fan of $e$, the coorientation on $\tau(2)$ totally orders the $\hat{\tau}$-faces incident to $e$ belonging to $\hat{\tau}$-tetrahedra in the fan. A normal curve $\gamma$ is a stable loop (unstable loop) if a lift $r\gamma$ to $\hat{\mathcal{M}}^\circ$ has the following property: at each $\hat{\tau}$-edge $e$ intersected by $\gamma$, $\gamma$ passes from a non-topmost (bottommost) face incident to $e$ to a topmost (second-bottommost) one. These curves were first studied in [Lan19]. Let $\gamma$ be a stable (unstable) loop and $\hat{\gamma}$ its lift to $\hat{\mathcal{M}}^\circ$. We say $\gamma$ is shallow if, whenever $\hat{\gamma}$ traverses a $\hat{\tau}$ edge, it passes from a second-topmost (bottommost) $\hat{\tau}$-face to a topmost (second-bottommost) $\hat{\tau}$-face.

Let $\gamma$ be a stable loop. Then by perturbing $\gamma$ slightly in the direction of the coorientation of $\tau(2)$, we can homotop $\gamma$ to a curve which is positively transverse to $\tau(2)$. Similarly if $\eta$ is an unstable loop, we can homotop $\eta$ slightly downward with respect to the coorientation to produce a closed positive transversal to $\tau(2)$. The images of $\gamma$ and $\eta$ under these homotopies are called the pushup and pushdown of $\gamma$ and $\eta$, respectively. See Figure 10.

If a normal curve is both a stable and unstable loop (up to switching orientation), we call it a normal branch loop. A normal branch loop can be pushed up to to a branch curve of $B^u$ and pushed down to a branch curve of $B^s$, and each branch curve of $B^u$ and $B^s$ is the image of such a homotopy. We record this fact in the lemma below; we encourage the reader to check it by drawing the intersection of $B^u$ and $B^s$ with the faces of all $\tau$-tetrahedra incident to a given $\tau$-edge (i.e. all the faces in a picture like Figure 7). Alternatively see [SS19, §6].

**Lemma 3.7** (Characterization of branch curves). Suppose $\gamma$ is a normal branch loop. Let $\gamma^+$ and $\gamma^-$ be the pushup and pushdown of $\gamma$, respectively. Then the $\Gamma$-cycles determined by
\( \gamma^+ \) and \( \gamma^- \) are branch curves of \( B^u \) and \( B^s \), respectively. Furthermore, every branch curve of \( B^s \) and \( B^u \) can be obtained this way.

Let \( \ell \) be a normal branch loop, and let \( e \) be a \( \tau \)-edge traversed by \( \ell \). Then the pushup of \( \ell \) passes through the \( \tau \)-tetrahedron for which \( e \) is the bottom \( \tau \)-edge, and also passes through each \( \tau \)-tetrahedron in one of the fans of \( e \). Symmetrically, the pushdown of \( \ell \) passes through each \( \tau \)-tetrahedron in the other fan of \( e \) and also the \( \tau \)-tetrahedron for which \( e \) is the top \( \tau \)-edge. Since every fan contains a hinge \( \tau \)-tetrahedron, the following is a consequence of Lemma 3.7:

**Lemma 3.8.** Every branch line of \( B^s \) or \( B^u \) passes through a hinge \( \tau \)-tetrahedron.

### 4. Flattened surfaces

#### 4.1. Conventions and preliminaries to flattening.

We now embark on the proof of Theorems A and B. We first need to define the notion of prongs from the theorem statement. Let \( t \) is a pseudohyperbolic train track on a torus \( T \), let \( \lambda \subset T \) be the union of all ladderpoles of \( t \), and let \( s \) is a slope on \( T \). Then we define

\[
\text{prongs}(s) := \frac{i_g(s, \lambda)}{2}
\]

where \( i_g \) is geometric intersection number. We say \( \text{prongs}(s) \) is the number of prongs of the slope \( s \).

For the remainder of the paper, except the final section, we will be in the following situation:

- \( M \) is a closed 3-manifold,
- \( U \) is a union of solid tori in \( M \),
- \( \tau \) is a veering triangulation of \( \tilde{M} := M - \text{int}(U) \), and
- the meridional slope for each component of \( U \) has \( \geq 3 \) prongs with respect to \( B_{\tau} \).

Hence we will think of \( \tilde{M} \) as a closed submanifold of \( M \), and \( \tau^{(2)} \) as a partial branched surface in \( M \).

**Lemma 4.1.** \( M \) is irreducible.

**Proof.** As is pointed out in [SS19, §6.5], \( B^s \) and \( B^u \) are laminar branched surfaces in \( M \) in the sense of Li. Hence [Li02, Theorem 1] gives that \( M \) contains an essential lamination and hence is irreducible by [GO89, Theorem 1]. □

In this section we discuss a process called flattening which takes an embedded surface and isotopes it so that \( S \cap \tilde{M} \) lies in a regular neighborhood of \( \tau^{(2)} \) in a controlled fashion. This will be a useful way to keep track of surfaces as we move them around in \( M \).

#### 4.2. The rod and plate neighborhood of \( \tau^{(2)} \).

We first construct a regular neighborhood of \( \tau^{(2)} \). For each \( \tau \)-edge \( e \), let \( R_e := e \times D^2 \) be the rod corresponding to \( e \) (where \( D^2 \) is a 2-dimensional disk). For a \( \tau \)-face \( \Delta \), let \( P_{\Delta} := \Delta \times [0, 1] \) be the plate corresponding to \( \Delta \). Then \( P_{\Delta} \) has a natural foliation \( \bigcup_{p \in \Delta} \{p\} \times [0, 1] \) by line segments that we call the vertical foliation of \( P_{\Delta} \). We orient the leaves of the vertical foliation so that, identifying \( \Delta \) with \( \Delta \times \{0\} \), the coorientation of \( \Delta \times \{0\} \) agrees with the orientation of the leaves.

We can glue up the rods and plates along their boundaries in a way prescribed by the identifications of \( \tau \)-edges and boundaries of \( \tau \)-faces to form the rod and plate neighborhood of \( \tau \), denoted \( N \). This gives a local picture as in the righthand side of Figure 11. We view \( N \) as embedded in \( M \subset \tilde{M} \), with \( \tau^{(2)} \) lying in its interior so that \( \tau^{(2)} \) intersects each leaf of the vertical foliation in each plate in exactly one point such that the vertical leaf orientations are compatible with the coorientation of \( \tau^{(2)} \). We call the union of the vertical foliations in
each plate the **vertical foliation** of $N_\epsilon$. Note that the rods are not foliated by the vertical foliation. There is a homotopy equivalence

$$\text{coll}: N_\epsilon \to \tau^{(2)}$$
called the **collapsing map** which is given by collapsing the disk factors of each rod and collapsing the vertical leaves in plates. This map restricted to $\tau^{(2)}$ is nearly the identity.

Let $\partial N_\epsilon = N_\epsilon \cap \partial M$. There is a natural decomposition of $\partial N_\epsilon$ into **junctions** and **conduits**, which are respectively the components of intersections of rods and plates with $\partial M$. The conduits inherit an oriented foliation from the vertical foliation, and the union of these foliations over all conduits is called the **vertical foliation** of $\partial N_\epsilon$. Similarly to $N_\epsilon$, we are leaving the junctions unfoliated.

Before beginning the flattening process, it is convenient to assume that each leaf of the vertical foliation that intersects $B^s$ is tangent to and contained in $B^s$. This can be achieved by a small isotopy.

**4.3. The flattening process.** Let $\Sigma$ be a sector of $\tilde{B}^s$. Because the underlying topology of $\tilde{B}^s$ is that of the dual 2-complex to $\tilde{\tau}$, $\Sigma$ is a topological disk pierced by a single $\tilde{\tau}$-edge $e$. Further, the intersection of $\tilde{\tau}^{(2)}$ with $\text{cl}(\Sigma)$ is a train track with a single switch at $e \cap \Sigma$ and one stop in each component of $\text{brloc}(\tilde{B}^s) \cap \text{cl}(\Sigma) - \{\text{triple points of } \tilde{B}^s\}$. See **Figure 12**.

Figure 12. The intersection of $\tau^{(2)}$ with a single sector of $B^s$. While the straight lines in the boundary of the sector denote branch segments of $B^s$, we do not use this additional information.
Let $S$ be a taut surface. By Lemma 2.2, we can assume that $S \cap B^s$ has simple patches.
Hence each patch is either a meridional disk, a nonmeridional disk, or an $\pi_1$-injective annulus
with respect to the tube in which it resides.

**Step 1.** We first perform an isotopy supported in a regular neighborhood of $B^s$ so that
$S \cap B^s$ lies in $N_\epsilon \cap B^s$ transverse to the induced vertical foliation. This can be done as
follows. Let $\Sigma$ be a sector of $B^s$, containing a single $B^s \cap \tau^{(2)}$-switch $v_\Sigma$. Since $S \cap B^s$ has
simple patches, $S \cap \Sigma$ contains no circle components and is thus a collection of arcs with
endpoints in $\partial \Sigma$. We can isotop $S$ so that each of these arcs lies in $N_\epsilon \cap \Sigma$ transverse to the
vertical foliation. This is shown in Figure 13. This isotopy can be performed consistently for
all sectors of $B^s$.

![Figure 13](image.png)  
*Figure 13. Step 1 of the flattening process in a single $B^s$-sector.  
We have not drawn the vertical foliation.*

**Step 2.** After step 1, $S$ lies close to $\tau^{(2)}$ near $B^s$. Consider a complementary region $R$ of
$B^s$. Because $B^s$ is dual to $\tau^{(2)}$ in $M$, $R$ is homeomorphic to an open solid torus containing
a single component $U_R$ of $U$. Let $T = \partial U_R$. Again by duality, we can identify $\tau^{(2)} \cap R$ with
$(\partial \tau^{(2)} \cap T) \times [0,1)$. Using this product structure, we can isotop $S$ so that $S \cap R \cap M$ lies
in $N_\epsilon \cap T$ transverse to the vertical foliation. This can be visualized as “combing” $S$ onto $\tau^{(2)}$
and into $U_R$. After performing this operation in all complementary regions of $B^s$, the
process is complete.

**Remark 4.2.** Note that we have broken symmetry slightly by flattening with respect to $B^s$.
We could also flatten with respect to $B^u$; the essential thing about $B^s$ that we use is its
duality with $\tau$.

4.4. **After flattening.** Now we reckon with the aftermath of the flattening process. We call
the resulting surface the flattening of $S$ and use the notation $S^0$. Any surface $F$ such that
each component of $F \cap U \pi_1$-injects into $U$ and $F \cap M$ is embedded in $N_\epsilon$ transverse to the
vertical foliation is called a flattened surface. A connected component of $S^0 \cap \{\text{plates of } N_\epsilon\}$
is called a **plate** of $S^0$. Similarly a connected component of $S^0 \cap \{\text{rods of } N_\epsilon\}$ is called a **rod**
of $S^0$. The total number of plates in $S^0$ is called the **area** of $S^0$. Let $\hat S^0 = S^0 \cap M$.

In each plate of $S^0$, the pairing of the coorientation of $S^0$ with the orientation of the vertical
foliation is either entirely positive or negative, and in these respective cases we call the plate
a **positive plate** or a **negative plate**. Each rod $r$ of $S^0$ is incident to two plates, and if both
plates are positive (negative) we say $r$ is a **positive (negative) rod**. If these two plates
have opposite sign we say $r$ is a **mixed rod**.
Let \( \text{neg}(S^q) \) be the subsurface of \( S^q \) obtained by taking the union of all negative plates and negative rods. Thus \( S^q \) is carried by the partial branched surface \( \tau^{(2)} \) exactly when \( \text{neg}(S^q) = \emptyset \). A detailed analysis of \( \text{neg}(S^q) \) for certain flattened surfaces will be central to our proof of Theorem 7.16.

We now describe the effects of flattening in relation to \( \partial \tau^{(2)} \).

- If \( p \) was a meridional disk patch of \( S \), then its image under flattening intersects \( U \) in a meridional disk whose boundary lies in \( \partial \mathcal{N} \) transverse to the vertical foliation.
- Similarly if \( p \) was an annulus patch of \( S \), then its image under flattening intersects \( U \) in an annulus whose boundary lies in \( \partial \mathcal{N} \) transverse to the vertical foliation, and whose core is homotopically nontrivial in \( U \).
- Finally, if \( p \) was a nonmeridional disk patch, then its image under flattening intersects \( U \) in a nonmeridional disk. The boundary of this nonmeridional disk is a component of \( \partial \tilde{\mathcal{M}} \) lying in \( \partial \mathcal{N} \) transverse to the vertical foliation, bounding a disk \( \delta \) in \( \partial U \). We will call \( \delta \) a \( b \)-disk of \( S^q \). A \( b \)-disk \( \delta \) is inward (outward) if the coorientation of \( S^q \) points into (out of) \( \delta \) along \( \partial \delta \). A \( b \)-disk is innermost if it contains no other \( b \)-disks in its interior.

Let \( \delta \) be a \( b \)-disk of \( S^q \). The volume of \( \delta \) is the number of components of \( \partial \tilde{\mathcal{M}} - \partial \mathcal{N} \) contained in \( \delta \). The circumference of \( \delta \) is the length of the image of \( \delta \) under the collapsing map \( \text{coll}_{\partial \mathcal{M}} : \partial \mathcal{N} \to \partial \tau^{(2)} \), with respect to a metric assigning length 1 to each \( \partial \tau^{(2)} \)-branch. Let \( T_\delta \) be the component of \( \partial \tilde{\mathcal{M}} \) containing \( \delta \). Let \( \tilde{\delta} \) be a lift of \( \delta \) to the universal cover \( \tilde{T}_\delta \) of \( T_\delta \). The width of \( \delta \) is the number of ladders in \( \tilde{T}_\delta \) such that the image of \( \partial \tilde{\delta} \) under collapsing intersects both ladderpoles.

The boundary of \( \delta \) decomposes into cooriented line segments contained alternately in junctions and conduits of \( \partial \mathcal{N} \), which we call junction segments and conduit segments of \( \partial \delta \), respectively. If there is no chance of confusion with junctions or conduits of \( \partial \mathcal{N} \), we will sometimes refer to a junction segment or conduit segment of \( \partial \delta \) as simply a junction or conduit of \( \delta \). A conduit segment is positive or negative if its coorientation respectively agrees or disagrees with the orientation of the vertical foliation. It is called a ladderpole or rung conduit segment if it intersects a ladderpole or rung of \( \partial \tau^{(2)} \), respectively. A junction segment is positive (negative) if connects two positive (negative) conduit segments, and mixed otherwise.

Usually when drawing a \( b \)-disk \( \delta \) we omit drawing the junctions and conduits of \( \partial \mathcal{N} \), simply drawing \( \partial \delta \) close to \( \partial \tau^{(2)} \) in such a way that it is clear which junctions and conduits \( \partial \delta \) traverses. See Figure 14.
Figure 15. We have drawn a junction (magenta) and six conduits (green) of $\partial N_e$. There are three cusps shown. On the left is a cusp of size 2. The top right cusp has size 0 and the bottom right cusp has size 1. The shading indicates the $b$-disk in whose boundary each cusp lies.

Figure 16. Face move: a size 0 cusp of an innermost $b$-disk for $S^9$ corresponds to a portion of $S^9$ running back and forth over a single $\tau$-face. This can be eliminated by an isotopy of $S^9$.

A cusp of $\delta$ is a mixed junction segment which is a “convex” part of $\partial \delta$. See Figure 15. To make this notion of convexity precise, let $c_1$ and $c_2$ be conduit segments of opposite sign connected by a junction segment $j$ contained in a junction $J$. Then near $J$, one of the $c_i$ is higher with respect to the coorientation of $\partial \tau(2)$; suppose that $c_1$ is higher. If the orientation of the vertical foliation points into $\delta$ along $c_1$, then $j$ is a cusp. Let $B_J$ be a small open neighborhood of $J$. Let $B_J(j)$ be the component of $\delta \cap B_J$ whose boundary in $B_J$ is $B_J \cap (c_1 \cup c_2 \cup j)$. The size of $j$ is the number of connected components of $(\partial \hat{M} - \partial N_e) \cap B_J(j)$.

We can use the orientation of the vertical foliation as well as the coorientation of $\partial \tau(2)$ to totally order all the conduits of $S^9$ meeting $J$ on the same side as $c_1$ and $c_2$. If there are no conduits of $\partial \hat{S}^9$ lying between $c_1$ and $c_2$ with respect to this order, we say the cusp $j$ is free.

4.5. Flat isotopy. Similar to our drawings of $\partial \hat{M}$, when drawing a flattened surface we omit drawing actual conduits and rods of $N_e$ and simply draw the surface near $\tau(2)$ in such a way that it is clear where $S^9$ lies in $N_e$.

Let $S^9$ be a flattened surface, whose identity will change by isotopies. We will now describe a series of possible isotopies we can perform on $S^9$. We draw pictures of these to aid the reader, but to reduce clutter we only will draw the parts of the isotopy that take place in $\hat{M}$.

**Face move.** Whenever $\partial \hat{S}^9$ has two conduit segments $c_1$ and $c_2$ connected by a junction $j$ such that $c_1$ is positive, $c_2$ is negative, and both $c_i$ lie in the same conduit, we say that the concatenation $c_1 * j * c_2$ is **backtracking**. This corresponds to a portion of $S^9$ running across
a plate corresponding to a $\tau$-face $\triangle$, into a rod and back across the same plate. If there are no junctions between $c_1$ and $c_2$, then this portion of $S^0$ can be eliminated by an isotopy of $S^0$ as in Figure 16. This induces an isotopy of $\partial S^0$ at two tips of the corresponding face $\triangle$, and to a cut and paste surgery of $\partial S^0$ at the third tip of $\triangle$. Let $U_1$ be the component of $U$ corresponding to this third tip. The cut and paste surgery is a result of pushing a small piece of $S^0$ which used to lie in $\bar{M}$ into $U_1$. If this creates a component of $S^0 \cap U_1$ which is not $\pi_1$-injective in $U$, we can use the irreducibility of $M$ and incompressibility of $S^0$ to correct this via an isotopy. The initial isotopy combined with the second (if necessary) is called a face move. We see that whenever $\partial S^0$ has backtracking, the area of $S^0$ can be reduced by at least 2 by a face move.

**Disk removals.** Let $\delta$ be an innermost $b$-disk. If $\delta$ has volume 1 and circumference 3 or volume 0 and circumference 2, then $\delta$ can be eliminated by an isotopy of $S^0$ so that $S^0 \cap M$ still lies in $N$, transverse to the vertical foliation. Each of these decreases the area of $S^0$ by 2. See Figure 17.

**Tetrahedron move.** Let $\delta$ be a $b$-disk, not necessarily innermost. If $c$ is a free cusp of $\delta$ with size 1, then there is an isotopy of $S^0$ as shown in Figure 18 which sweeps a portion of $S^0$ across a $\tau$-tetrahedron. This move decreases the volume of $\delta$, and if $\delta$ is inward (outward) it does not increase the volume of any inward (outward) $b$-disk.
**Width one move.** If \( \delta \) has width 1 and no backtracking, then \( \delta \) has a cusp of size 0; this fact follows from Lemma 3.2. If \( \delta \) has volume \( \geq 2 \), then after applying the corresponding tetrahedron move to this cusp, the result is another 5-disk of width 1 with no backtracking. Hence we can eventually shrink the 5-disk to one with volume 1 and perimeter 3, to which we can apply a disk removal. It follows that if \( S^5 \) has a 5-disk with width 1.

**Flip move.** If \( S^5 \) has two adjacent plate regions \( P \) and \( Q \) corresponding to the bottom of a single \( \tau \)-tetrahedron \( \otimes \), and \( P \) and \( Q \) are topmost in their respective plates of \( N_\tau \), then there is an isotopy of \( S^5 \) which sweeps a portion of \( S \) up through \( \otimes \) and replaces \( P \) and \( Q \) by two bottommost plate regions \( P' \) and \( Q' \) in the plates corresponding to the top of \( \otimes \). This is called an upward flip move. The inverse of an upward flip is called a downward flip move.

**Annulus move.** Our last move is different than the previous ones in that its nature is more global than local. Suppose there is an annulus \( A \subset S^5 \) with the following properties:

1. \( A \) is a union of rods and plates
2. each plate of \( A \) is topmost in its \( N_\tau \)-plate, and
3. the induced stable train track on \( A \) carries a curve whose image under collapsing is a shallow stable loop.

Then we can perform a stable annulus move, which we now describe. Each plate region \( \rho_i \) corresponds to a \( \tau \)-face \( \Delta_i \), and each \( \Delta_i \) is a bottom \( \tau \)-face for a \( \tau \)-tetrahedron \( \otimes_i \). There is an isotopy of \( S^5 \) which sweeps \( A \) up through \( \bigcup_i \otimes_i \), pushing the core of \( A \) into a component \( U_A \) of \( U \). This move leaves the area of \( S^5 \) unchanged and introduces two new boundary components of \( S^5 \) which traverse only ladderpole conduits. See Figure 20.

An unstable annulus move is defined symmetrically. For this move we require each plate region of the annulus \( A \) to be bottommost in its plate, and we require that the induced unstable train track carry a curve which collapses to a shallow unstable loop.

5. \( C^\gamma \subset \text{cone}(\sigma_\tau) \)

Let \( C_\tau \subset H_1(M) \), the cone of homology directions of \( \tau \), be the smallest convex cone in \( H_1(M) \) containing the homology classes of oriented cycles in \( \Gamma \). Associated to \( C_\tau \) is a dual cone in \( H_2(M) \) defined by

\[
C^\gamma = \{ \alpha \in H_2(M) \mid \langle \alpha, \gamma \rangle \geq 0 \text{ for all } \gamma \in C_\tau \}.
\]

Let \( U_i \) be a component of \( U \) and let \( \text{prongs}(U_i) \) be the number of prongs of the meridional slope of \( U_i \). The component \( R_i \) of \( M - B^* \) containing \( U_i \) is homeomorphic to an open solid torus. Lemma 3.6 gives us that its closure \( C_i \) in the path topology has the smooth structure of the mapping torus of some rotation of a \( (\text{prongs}(U_i)) \)-gon patch of a train track. We call
Figure 20. An example of a stable annulus move. Left: a portion of a veering triangulation. Some edge identifications are specified, but we make no assumptions about the others. The portion of $B^s \cap \tau^{(2)}$ shown carries a shallow stable loop. Center: An example of an annulus satisfying the conditions for a stable annulus move. Right: the situation after the stable annulus move. The core of the annulus has been pushed into the component of $U$ corresponding to the topmost (in the page) tips of the four $\tau$-tetrahedra.

$R$ a tube of $B^s$. Similarly $U_i$ determines a complementary region of $B^u$ with the same type of structure, which we call a tube of $B^u$. The index of $U_i$ is the number

$$\text{index}(U_i) = 2 - \text{prongs}(U_i),$$

which is equal to the minimal index of a meridional disk of $C_i$. Since $\text{prongs}(U_i) \geq 3$, we have $\text{index}(U_i) \leq -1$.

**Lemma 5.1** (no nullgons or monogons). Let $S$ be an incompressible surface in $M$. Then $S$ is isotopic to a flattened surface $S^\flat$ such that all patches of both $S^\flat \cap B^s$ and $S^\flat \cap B^u$ are simple and have nonpositive index.

**Proof.** We first isotop $S$ so that it has simple patches. Then we flatten $S$ to a surface $S^\flat$. Let $t^s$ and $t^u$ denote $S^\flat \cap B^s$ and $S^\flat \cap B^u$. Any meridional disk patch of $t^s$ or $t^u$ has index $\leq -1$ because each component of $U$ has index $\leq -1$. By applying face moves and width one moves to decrease the area of $S^\flat$, we can isotop $S^\flat$ so that it has no $b$-disks of width one or zero. Suppose that $p$ is a nonmeridional disk patch of $t^s$. If $\delta$ has width $\geq 2$, $\partial \delta$ must contain at least two rung segments crossing an upward ladder, so by **Lemma 3.6** $p$ must have at least 2 switches. Symmetric reasoning shows the same holds for patches of $t^u$.

In summary, every disk patch has nonpositive index. Since topological annulus patches must have nonpositive index, the proof is complete. \qed

**Lemma 5.2** ($M$ is atoroidal). There is no incompressible torus embedded in $M$.

**Proof.** Suppose for a contradiction that $T$ is an incompressible torus embedded in $M$. By **Lemma 5.1**, $T$ can be flattened to $T^\flat$ such that $T^\flat \cap B^s$ has no patches of positive index. Since $\chi(T) = 0$, all patches of $T^\flat \cap B^s$ must have index 0. Therefore, since the index of each tube is at least $-1$, $T^\flat$ must not intersect any tube in a meridional disk. As such we may isotop $T^\flat$ so that it lies in $M$. By [HRST11, Theorem 1.5] $\tau$ admits a strict angle structure and by a result of Casson exposited in [FG11, Theorem 1.1], $\text{int}(M)$ is irreducible and atoroidal. In particular $T^\flat$ must be either peripheral in $M$, in which case $T$ is compressible in $M$, a contradiction; or compressible in $M$, in which case $T$ is also compressible in $M$. \qed
It follows that we can speak of the Thurston norm $x$ and the dual norm $x^*$ as norms on $H_2(M)$ and $H_1(M)$, respectively. The Euler class of $\tau$ is the homology class in $H_1(M)$ defined by

$$e_\tau = \frac{1}{2} \cdot \sum \text{index}(U_i) \cdot [\text{core}(U_i)],$$

where the sum is over components of $U$ and core($U_i$) denotes the core curve of $U_i$, oriented so that a curve in $\partial U_i$ lying in a single ladder positively transverse to its rungs is homotopic to a positive multiple of core($U_i$). We define

$$E = \{ \alpha \in H_2(M) \mid \langle e_\tau, \alpha \rangle = x(\alpha) \}.$$

By definition, if $E$ is strictly larger than $\{0\}$ then it is a cone over some face of $B_x(M)$. We denote this face by $\sigma_\tau$. If $E = \{0\}$ we let $\sigma_\tau$ be the empty face of $B_x(M)$.

**Lemma 5.3.** The Euler class of $\tau$ has dual Thurston norm at most 1:

$$x^*(e_\tau) \leq 1.$$

**Proof.** It suffices to show that for any taut surface $S$ we have $|\langle e_\tau, [S] \rangle| \leq \chi_-(S)$ or alternatively that $\langle e_\tau, [S] \rangle \geq \chi(S)$. First, we can flatten $S$ to $S^\#$ and eliminate $b$-disks of width $\leq 1$, guaranteeing that no patches of $S^\# \cap B^\#$ have positive index (the same is true for $S^\# \cap B^\#$ but we do not need that here). The only patches which have nonzero intersection with $\bigcup_{i} \text{core}(U_i)$ are the meridional disk patches. If $C$ is a meridional disk patch, let sign($C$) = $\pm 1$ according to whether the algebraic intersection of $C$ with the core of the corresponding component of $U(C)$ of $U$ is $\pm 1$. Then

$$2\langle e_\tau, [S] \rangle = \sum_{\text{merid}} \text{sign}(C) \cdot \text{index}(U(C)),$$

where the sum is over only meridional patches. Since all patches have nonpositive index,

$$\sum_{\text{merid}} \text{sign}(C) \cdot \text{index}(U(C)) \geq \sum_{\text{merid}} \text{index}(C) \geq \sum_{\text{all patches}} \text{index}(P) = 2\chi(S^\#) = 2\chi(S),$$

where the final sum is over all patches of $S^\# \cap B^\#$. \hfill $\square$

**Lemma 5.4.** If $S$ is carried by $\tau^{(2)}$ (as a partial branched surface) then $S$ is taut and $[S] \in \text{cone}(\sigma_\tau)$.

**Proof.** First we observe that for each $\tau$-face $\triangle$, there is an oriented closed curve through $\triangle$. This follows from the fact that the dual graph $\Gamma$ is strongly connected, which we leave to the reader (hint: any connected digraph such that each vertex has the same incoming and outgoing valence is strongly connected). Hence any component of a surface carried by $\tau^{(2)}$ pairs positively with a class in $H_1(M)$, and is not nullhomologous.

Next we show that $\chi_-(S) = x([S])$. We have $x([S]) \leq \chi_-(S)$ by the definition of $x$, and by the above lemma we have that $x^*(-e_\phi) \leq 1$. If $S$ is carried by $\tau^{(2)}$ then each patch of $S^\# \cap B^\#$ intersecting a component $u$ of $U$ corresponds to either a meridional curve on $\partial U$ which traverses exactly prongs($u$) rungs of upward ladders, or to 2 ladderpole curves crossing no rungs. It follows that each patch of $S \cap B^\#$ with nonzero index is a meridional disk patch with index equal to the index of the corresponding tube. Hence $\langle e_\tau, [S] \rangle = \chi(S)$. Therefore

$$\chi_-(S) = \langle -e_\phi, [S] \rangle \leq x([S]),$$

completing the proof. \hfill $\square$
**Theorem 5.5.** Let \( \alpha \in H_2(M) \) be an integral class. Then \( \alpha \in C^\vee_\tau \) if and only if there exists a surface \( S \), necessarily taut, carried by \( \tau^{(2)} \) with \( [S] = \alpha \).

**Proof.** If \( S \) is carried by \( \tau \) and represents \( \alpha \), it is clear that \( \alpha \in C^\vee_\tau \).

Conversely, suppose \( \alpha \in C^\vee_\tau \). Let \( u \in H^1(M) \) be the Poincaré dual of \( \alpha \). Note that \( u \) pulls back to a cohomology class \( \hat{u} \) on \( \hat{M} \) which takes integral values on directed cycles in \( \Gamma \). Let \( \hat{\alpha} \in H_2(M, \partial \hat{M}) \) be the Lefschetz dual of \( \hat{u} \). As explained in [LMT20, Proposition 5.11], \( \alpha \) is represented by a surface \( \hat{S} \) which is carried by \( \tau^{(2)} \), viewed as a branched surface in \( \hat{M} \).

The proof of [Lan18, Lemma 3.3] goes through in this setting and thus, for each component \( \partial U_i \) of \( \partial M \), \( S \cap \partial U_i \) is either (i) a nullhomologous collection of curves of ladderpole slope on \( \partial U_i \), (ii) a collection of curves with the meridional slope of \( U_i \), or (iii) empty. Hence \( \hat{S} \) can be capped off to give a surface \( \hat{S} \subset M \) which is carried by \( \tau^{(2)} \). The composition \( P \) of the maps

\[
H_2(M) \xrightarrow{PD} H^1(M) \xrightarrow{i^*} H^1(\hat{M}) \xrightarrow{LD} H_2(\hat{M}, \partial \hat{M}),
\]

where the first and last arrows are Poincaré and Lefschetz duality and the middle arrow is pullback under inclusion, is injective as explained in [Lan18, §2.1] (in that paper it is called the puncturing map). We have \( P(\alpha) = \hat{\alpha} = P([S]) \), so \( [S] = \alpha \). By Lemma 5.4, \( S \) is taut.

**Corollary 5.6.** \( C^\vee_\tau \subseteq \text{cone}(\sigma_\tau) \).

**Proof.** If \( \alpha \in C^\vee_\tau \) is an integral class then Theorem 5.5 furnishes a surface representing \( \alpha \) carried by \( \tau^{(2)} \), so \( \alpha \in \text{cone}(\sigma_\tau) \) by Lemma 5.4.

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### 6. The bigon property and efficient bigon property

#### 6.1. The bigon property

Let \( S \) be an embedded surface in \( M \) transverse to \( B^s \) and \( B^u \). Each switch of \( S \cap B^s \) corresponds to an intersection of \( S \) with \( \Gamma \), and such an intersection point can be either positively or negatively oriented according to whether the orientation of \( \Gamma \) agrees with the coorientation of \( S \). A point can be either positively or negatively oriented according to whether the orientation of \( S \) agrees with the coorientation of \( \Gamma \).

A switch of \( S \cap B^s \) may be either positive or negative. A positive switch increases \( \chi \), and a negative switch decreases \( \chi \).

A switch of \( S \cap B^s \) may be either positive or negative. A positive switch increases \( \chi \), and a negative switch decreases \( \chi \).

Then we say \( S \) has the **stable bigon property**. If \( S \) has the above properties with respect to \( B^u \), then \( S \) has the **unstable bigon property**. If \( S \) both the stable and unstable bigon properties, we say \( S \) has the **bigon property**. The bigon property will provide us with traction as we try to isotop surfaces to be carried by \( \tau^{(2)} \). Although we will not need to use the following fact, the bigon property has topological and algebraic consequences:

**Lemma 6.1.** Let \( S \) be a surface with the stable or unstable bigon property. Then \( S \) is taut and \( [S] \in \text{cone}(\sigma_\tau) \).

**Proof.** We break symmetry and suppose that \( S \) has the stable bigon property. Then the only patches of \( S \cap B^s \) which possibly have nonzero index are meridional disk and topological annulus patches. It is impossible for a topological annulus patch of \( S \cap B^s \) to have a positive switch without also having a negative switch (this follows from the structure of the tubes of \( B^s \) and \( B^u \), see discussion at the beginning of Section 5), so the topological annulus patches must be annulus patches i.e. must have no switches. Hence \( 2\chi(S) \) is the sum of the indices
of its meridional patches. Because the switches of these patches are all positive, each one contributes +1 to the algebraic intersection of \([S]\) with the core of the tube. Hence \(2\chi(S)\) is equal to \(2\langle e_\tau, [S]\rangle\), so

\[
\chi_-(S) = \langle -e_\tau, [S]\rangle \leq x([S]) \leq \chi_-(S),
\]

using the fact that \(x^*(-e_\tau) \leq 1\) by Lemma 5.3. Since no component of \(S\) is nullhomologous, \(S\) is taut.

**Lemma 6.2.** Let \(S\) be a taut surface in \(M\) with \([S] \in \text{cone}(\sigma_\tau)\). Then \(S\) is isotopic to a flattened surface with the bigon property.

**Proof.** By Lemma 5.1, we can flatten \(S\) to \(S^6\) so that \(S^6\) has simple patches with nonpositive index. We will show that \(S^6\) has the stable bigon property; the proof that \(S^6\) has the unstable bigon property is entirely similar.

To prove \(S^6\) has the stable bigon property it suffices to show that any patch of \(S^6 \cap B^6\) which is not a bigon has only positive switches.

For a tube \(T\) of \(B^6\), define index\((T)\) to be the index of the component of \(U\) at the core of \(T\). Just as in the proof of Lemma 5.3, the patches of \(S^6\) are all topologically disks or annuli and of these, the only patches having nonzero algebraic intersection with the core of the corresponding tube are meridional disks. Therefore the definition of \(e_\tau\) gives us

\[
2\chi(S^6) = -2x([S^6]) = 2e_\varphi([S^6]) = \sum_{\text{tubes } T} \left( \sum_{\text{merid. disks } D} \text{index}(T) \cdot \text{sign}(D) \right),
\]

where \(\text{sign}(D) = \pm 1\) is the algebraic intersection of \(D\) with the core of the corresponding tube. Subtracting Equation (2.1) from the above, we have

\[
0 = \sum_{\text{tubes } T} \left( \sum_{\text{merid. disks } D} \text{index}(T) \cdot \text{sign}(D) - \text{index}(D) \right) - \sum_{\text{nonmerid. } C} \text{index}(C),
\]

so

\[
\sum_C \text{index}(C) = \sum_T \left( \sum_D \text{index}(T) \cdot \text{sign}(D) - \text{index}(D) \right).
\]

Every term of the lefthand sum is nonpositive since there are no monogons or nullgons. Moreover every term of the righthand sum is nonnegative. For disks with \(\text{sign}(D) = -1\) this is trivial and for disks \(D\) with \(\text{sign}(D) = 1\) this follows from the fact that any meridional disk of a tube \(T\) has index bounded above by the index of the tube in which it sits.

We conclude that each term is zero. It follows that every nonmeridional patch has index 0 and is therefore an annulus or bigon. Each meridional patch has sign 1 and index equal to the index of the corresponding tube, so each switch of a non-bigon patch must be positive, whence \(S\) has the stable bigon property. An identical argument shows that \(S^6\) has the unstable bigon property. \(\square\)

**Lemma 6.3.** Let \(S^\circ\) be a flattened surface with the bigon property and let \(\gamma\) be a component of \(S^\circ \cap \partial M\). Then the following are equivalent:

1. \(\gamma\) contains a negative rung conduit,
2. \(\gamma\) bounds a \(b\)-disk, and
3. the patches of \(S^\circ \cap B^\circ\) and \(S^\circ \cap B^a\) containing \(\gamma\) are both bigons.

**Proof.** This follows from Lemma 3.6 and the definition of the bigon property. \(\square\)
Lemma 6.4. Let $S^5$ be a flattened surface with the bigon property. Each $b$-disk $\delta$ of $S^5$ has width two and has exactly four rung conduit segments. If in addition $S^5$ has a $b$-disk with two adjacent rung conduit segments meeting at a free cusp of size 1, then $S^5$ is flat isotopic to a surface with smaller area.

Proof. Let $t^s = S^5 \cap B^s$ and $t^u = S^5 \cap B^u$. The $b$-disk $\delta$ corresponds to a bigon patch of $t^s$, so $\partial \delta$ must contain exactly two rung conduit segments crossing upward ladders by Lemma 3.6. Similarly $\delta$ corresponds to a bigon patch of $t^u$, so $\partial \delta$ must contain exactly two rung conduit segments crossing downward ladders.

For the last claim, note that performing the available tetrahedron move on the free cusp creates a $b$-disk of width one. Applying a subsequent width one move decreases the area of $S^5$. □

Corollary 6.5. Suppose $S^5$ is a flattened surface with the bigon property. Suppose that $S^5$ has two plates $P_1$ and $P_2$ separated by a single rod such that there exists a hinge tetrahedron $\bigotimes$ which is immediately above $P_1$ and below $P_2$. Then $S^5$ is flat isotopic to a surface with smaller area.

Proof. There is a unique tip $\triangle$ of $\bigotimes$ such that the corresponding tips $t_1$ and $t_2$ of $P_1$ and $P_2$ are both rung conduit segments (see Figure 21). The signs of $t_1$ and $t_2$ are opposite, so $t_1$ and $t_2$ must lie in the boundary of a $b$-disk by Lemma 6.3. Since each $b$-disk has width two and four rung conduit segments in its boundary, $t_1$ and $t_2$ must meet at a free cusp of size 1. Now we apply the last sentence of Lemma 6.4. □

Lemma 6.6. Suppose that $S^5$ is a flattened surface with the bigon property and that $S^5$ has a $b$-disk with a free cusp of size 1. After performing the associated tetrahedron move, $S^5$ either still has the bigon property or is flat isotopic to a surface with smaller area.

Proof. A tetrahedron move does not change the underlying topology of the patches of $S^5 \cap B^s$ or $S^5 \cap B^u$. Suppose without loss of generality that that a meridional or topological annulus patch $p$ of $S^5 \cap B^s$ has a negative switch after a tetrahedron move. Then the index of $p$ is strictly less than it was prior to the tetrahedron move. Since the sum of indices over all patches remains unchanged, the index of some patch $q$ must have increased. Since $S^5$ initially had the bigon property, $q$ must be a nonmeridional disk patch. Therefore $q$ must have positive index, i.e. be a nullgon or monogon, after the tetrahedron move. Since a $b$-disk cannot contain a single rung conduit, by Lemma 3.6 $q$ must be a nullgon and correspond to a $b$-disk of width 1. Now applying a width one move decreases the area of $S^5$. □
Figure 22. An inefficient $b$-disk with 2 kinks shown in magenta: $\kappa_1$ is an upper kink of length 2 and $\kappa_2$ is a lower kink of length 1. In cyan we see shadow $p\kappa_1q$, and shadow $p\kappa_2q$ is shown in green.

6.2. The efficient bigon property. Let $S^9$ be a flattened surface with the bigon property. Then there are limitations on the types of $b$-disks that $S^9$ can have, described in Lemma 6.4: the $b$-disks of $S^9$ all have width 2 and their boundaries contain four rungs. We wish to now simplify this picture by placing further restrictions on $b$-disks.

If $\delta$ is a $b$-disk of $S^9$ such that negative conduits are unlinked from positive conduits in $\partial\delta$, we say $\delta$ is efficient. The obstruction to the efficiency of $\delta$ is the existence of a kink in $\partial\delta$, which is a portion $k$ of $\partial\delta$ containing only ladderpole conduits of the same sign $s$ such that the conduits to either side of $k$ are of the same sign $s'$ with $s \neq s'$. See Figure 22.

Let $\delta$ be a $b$-disk of $S^9$ and let $r_1$ and $r_2$ be two rung conduits of $\partial\delta$ contained in the same ladder $L$. If there is an oriented transversal to $\partial\tau^{(2)}$ contained in $L \cap \delta$ starting at $r_1$ and ending at $r_2$, we say $r_1$ is a lower rung for $\partial\delta$ and $r_2$ is an upper rung for $\partial\delta$. A kink $\kappa$ always connects two upper rungs or two lower rungs. If $\kappa$ connects two upper or lower rungs, we say $\kappa$ is an upper or lower kink, respectively. The length of a kink $k$ is the number of conduits it contains.

Let $\kappa$ be a kink in $\partial\delta$ for a $b$-disk $\delta$ of $S^9$. Then $\kappa$ determines a region of $\delta$ called the shadow of $\kappa$, which we now define. If $\kappa$ is an upper (lower) kink, let $L$ be downward (upward) ladder incident to $\kappa$. Let $v$ be the lowest (highest) junction traversed by $\kappa$. Let $r$ be the upper (lower) rung in $\partial\delta$ crossing $L$, and let $r'$ be the topmost (bottommost) rung in $L$ which is incident to $v$. The region in $L$ bounded above (below) by $r$ and below (above) by $r'$ is called the shadow of $\kappa$ and denoted shadow($\kappa$). See Figure 22.

Definition 6.7. Let $S^9$ be a flattened surface with the bigon property. We say that $S^9$ has the efficient bigon property if $\partial\delta S^9$ has no backtracking and every $b$-disk of $S^9$ is efficient.

Our next goal is to show that by applying flat isotopies to decrease the area of $S^9$, we can assume that it has the efficient bigon property.

Lemma 6.8. Let $S^9$ be a flattened surface with the bigon property. Let $\delta$ be an inefficient $b$-disk with kink $\kappa$. If the shadow of $\kappa$ contains a hinge flat triangle, then $S^9$ is flat isotopic to a surface with smaller area.

Proof. There is a portion of $\partial\delta$ that, omitting junctions, can be expressed as a concatenation $(r_1, \kappa, r_2, s)$ where $r_1$ and $r_2$ are rung conduits of the same sign, and there is a cusp at the junction of $r_2$ with $\kappa$ and $s$ ($s$ could be rung or ladderpole in this discussion). Let $L_i$ be the
ladder crossed by $r_i$, $i = 1, 2$. We assume that $\kappa$ is an upper kink; the argument when $\kappa$ is a lower kink is symmetric. We have drawn this picture in Figure 23.

Suppose that $\text{shadow}(\kappa)$ contains a hinge flat triangle $h$. We will perform tetrahedron moves until we create a $b$-disk of width one which we can eliminate, reducing the number of $b$-disks by one. If $\delta$ is innermost, by applying tetrahedron moves we can begin shrinking $A$ one flat triangle at a time. After each move, by Lemma 6.6 $S^0$ either has the bigon property or has a $b$-disk of width 1. If none of these moves create a $b$-disk of width 1, we eventually arrive in a position where there is an available tetrahedron move sweeping $\partial \delta$ across $h$. By Corollary 6.5, this creates a $b$-disk of width 1 and $S^0$ is flat isotopic to a surface with smaller area.

If $\delta$ is not innermost, note that any $\delta$ disk containing $h$ and contained in $\delta$ must contain $h$ in the shadow of a kink. Hence by passing to a $b$-disk contained in $\delta$ we can assume that $\delta$ contains $h$ in the shadow of a kink and that no $b$-disks contained in $\delta$ contain $h$. Now we shrink $\delta$ as before, checking after each tetrahedron move to see if we have created any width 1 $b$-disks. In shrinking $\delta$, it is possible that $b$-disks of the opposite coorientation contained in $\delta$ may grow. If some such disk grows so that it contains $h$, then we are done by Corollary 6.5. Otherwise we can eventually move $\partial \delta$ across $h$ and Corollary 6.5 again finishes the proof. □

**Lemma 6.9.** Let $S^0$ be a flattened surface with the bigon property. If $S^0$ does not have the efficient bigon property, then $S^0$ is flat isotopic to a surface with smaller area.

*Proof.* Suppose $\delta$ is an inefficient $b$-disk of $S^0$ with kink $\kappa$. By Lemma 6.8, we can assume $\text{shadow}(\kappa)$ contains only non-hinge flat triangles.

Let $r_1, r_2, s, L_1, L_2$ be defined as in the previous lemma. We assume that $L_1$ is an upward ladder and $L_2$ is downward; the argument in the other case will be symmetric. We may assume the downward flat triangle in $L_2$ immediately below shadow($\kappa$) is non-hinge. For suppose otherwise; then after a sequence of tetrahedron moves there would be an available tetrahedron move across the hinge tetrahedron and we would be finished by Corollary 6.5. We have drawn the situation in Figure 24, in the case where $\kappa$ has length 2.

In Figure 25, we have drawn the tetrahedra corresponding to the three flat triangles shown in Figure 24. Let $a, r_2', s'$ be the conduits of $S^0 \cap \partial M$ as labeled in Figure 25. Note that $s'$ is a rung conduit since it is a 0-0 branch of an upward flat triangle. Also, since all the flat triangles in $\text{shadow}(\kappa)$ are non-hinge, $r_2'$ is a rung conduit. We conclude that the junction of $r_2'$ and $s'$ must be a cusp of a $b$-disk $\delta'$, by Lemma 6.3.
We may assume that the next edge in $\partial \delta'$ in the sequence $(s', r_2', \ldots)$ is not $a$. If it were, then $S^0 \cap \partial \hat{M}$ would wrap around 3 conduits of the leftmost flat triangle in Figure 25 (colored magenta), and after a tetrahedron move $\partial \delta^b$ would have backtracking and a face move would decrease the area of $S^0$. For clarity, we first assume that $\delta$ is innermost. In this case we can perform tetrahedron moves to shrink shadow $(\kappa)$ until $\kappa$ no longer exists. These moves either introduce a new kink to $\delta'$, or lengthen a kink on $\delta'$ (by 2, in the case shown in Figure 25). Let $\kappa'$ be this next kink. By Lemma 6.8 we can assume shadow($\kappa'$) consists of only of non-hinge flat triangles. We call the bottommost one $t'$. We perform a flat isotopy to eliminate this new kink, and so on. This gives a sequence of non-hinge tetrahedra $(\otimes, \otimes', \otimes'', \ldots)$ corresponding to the sequence $(t, t', t'', \ldots)$ of flat triangles. This gives a path in $\Gamma$ which can be pushed up into $\tau^{(2)}$ to a normal curve traversing $\tau$-edges from bottommost to topmost faces. By Lemma 3.7, this defines a branch line. By Lemma 3.8, some $t^{(i)}$ will eventually be hinge, allowing us to reduce the area of $S^0$ by Lemma 6.8 and completing the proof in this case.
Now suppose that \( \delta \) is not innermost. Let \( \gamma(0) \) be the \( \Gamma \)-vertex in the \( \tau \)-tetrahedron corresponding to \( t \), where as before \( t \) is the lowest flat triangle in \( \text{shadow}(\kappa) \). Let \( \gamma(-1) \) be the \( \Gamma \)-vertex inside the \( \tau \)-tetrahedron corresponding to the upward flat triangle sharing a ladderpole branch with \( t \). Let \( \gamma \) be the stable branch line in \( \Gamma \) determined by the \( \Gamma \)-edge \((\gamma(-1), \gamma(0))\). This defines a sequence of \( \Gamma \)-vertices \((\gamma(0), \gamma(1), \gamma(2), \ldots)\). Let \( \bigoplus_i \) be the \( \tau \)-tetrahedron corresponding to \( \gamma(i) \).

By passing to a \( b \)-disk contained in \( \delta \) we can assume \( \delta \) is innermost in \( \text{shadow}(\kappa) \), meaning that \( \delta \) intersects \( \text{shadow}(\kappa) \) and does not contain any \( b \)-disks intersecting \( \text{shadow}(\kappa) \). Note that even if we pass to an inner \( b \)-disk, we do not relabel \( \kappa \).

We now explain how to perform a flat isotopy resulting in a situation where \( S^b \) has an inefficient \( b \)-disk \( \delta_1 \) with a kink \( \kappa_1 \) such that \( \text{shadow}(\kappa_1) \) contains a tip of \( \bigoplus_1 \). Shrink \( \delta \) by tetrahedron moves until the first time that either \( \delta \) or a \( b \)-disk contained in \( \delta \) has an innermost size 1 cusp at \( t \). Then as in the innermost case above, after performing a tetrahedron move on this cusp there is an inefficient \( b \)-disk with a kink \( \kappa_1 \), such that the tip of \( \bigoplus_1 \) is the lowermost flat triangle in its shadow. We can now iterate this process. Since some \( \bigoplus_i \) is eventually hinge, the lemma is proved by Lemma 6.8. \( \square \)

7. Finishing the proof of Theorems A and B

7.1. Efficient \( b \)-disks. Let \( S^b \) be a flattened surface. Let \( r \) be a mixed rod of \( S^b \) which is incident to a positive plate \( P \) and a negative plate \( Q \) and corresponds to a \( \tau \)-edge \( e \). If \( P \) lies above \( Q \) at \( e \) with respect to the coorientation on \( \tau(2) \) then we say \( S^b \) **curls down at** \( r \) and otherwise we say \( S^b \) **curls up at** \( r \).

**Lemma 7.1.** Let \( S^b \) be a flattened surface such that \( \partial S^b \) has no backtracking, and let \( Q \) be a negative plate of \( S^b \) incident to a rod \( r \). If the switch of \( Q \cap B^u \) points toward \( r \), then \( S^b \) does not curl down at \( r \). If the switch of \( P \cap B^u \) points toward \( r \), then \( S^b \) does not curl up at \( r \).

**Proof.** Let \( \Delta \) be the \( \tau \)-face corresponding to \( Q \), and let \( e \) be the \( \tau \)-edge corresponding to \( r \). If the switch of \( P \cap B^u \) points toward \( r \), then \( e \) is the bottom \( \tau \)-edge of the \( \tau \)-tetrahedron for which \( \Delta \) is a bottom \( \tau \)-face, so \( S^b \) cannot curl down at \( r \) without backtracking. The other case is symmetric. \( \square \)

We now develop some terminology to discuss efficient \( b \)-disks. An efficient \( b \)-disk \( \delta \) has 2 cusps whose complement in \( \partial \delta \) has two components which we will distinguish in two different ways: top/bottom and positive/negative. The **top** (bottom) of \( \delta \) is the component of \( \partial \delta \) along which the coorientation of \( \partial \tau \) points out of (into) \( \delta \).

One of these pieces contains only negative conduit segments and the other contains only positive conduit segments. We call the piece containing the negative conduit segments the **negative boundary** of \( \delta \), denoted \( \delta^- \). The other piece is the **positive boundary** of \( \delta \), denoted \( \delta^+ \). Note that the top of an outward \( b \)-disk is positive, and the top of an inward \( b \)-disk is negative.

Using this new language, we record a lemma describing the curling of \( S^b \) at the boundary of \( \text{neg}(S^b) \), which as a reminder is the union of all negative plates and rods of \( S^b \).

**Lemma 7.2.** Let \( S^b \) be a surface with the efficient bigon property. Let \( \gamma \) be a component of \( \text{neg}(S^b) \cap \partial M \) which is homeomorphic to an interval and contains a rung conduit segment. Let \( r_1 \) and \( r_2 \) be the rods of \( S^b \) corresponding to the junctions incident to the endpoints of \( \gamma \). Then \( S^b \) either curls up at both \( r_i \) or down at both \( r_i \).
Proof. Because \( S^b \) has the efficient bigon property, \( \gamma \) is equal to the negative boundary of some \( b \)-disk \( \delta \). If \( \delta \) is an outward \( b \)-disk, then \( \gamma \) is equal to the bottom of \( \delta \) and \( S^b \) must curl downward at both \( r_i \). Symmetrically, if \( \delta \) is inward then \( \gamma \) is the top of \( \delta \) and \( S^b \) must curl upward at both \( r_i \).

7.2. The excellent bigon property. Let \( S^b \) be a flattened surface with the efficient bigon property. Note that \( \text{neg}(S^b) \) naturally has the structure of a surface with corners. There are 2 corners for each interval component of \( \text{neg}(S^b) \cap \partial U \), and we can smooth the boundary of \( \text{neg}(S^b) \) by extending \( \text{neg}(S^b) \) slightly into \( U \) between each pair of corners as in Figure 26. Denote this smoothing of \( \text{neg}(S^b) \) by \( N \). The reason for forming \( N \) is that we will be interested in the indices of patches of certain train tracks on \( \text{neg}(S^b) \) where we ignore the corners of \( \text{neg}(S^b) \) but include the corners corresponding to train track stops.

Lemma 7.3. No patch of \( N \cap B^s \) or \( N \cap B^u \) has negative index. If \( p \) is an annulus patch of \( N \cap B^s \) or \( N \cap B^u \), then \( p \) has no switches and \( p \cap \partial S^b \) contains only ladderpole conduit segments.

Proof. Let \( N^s = N \cap B^s \). Each patch of \( N^s \) is either a topological annulus with no corners or a topological disk with two corners.

Suppose some patch \( p \) of \( N^s \) has negative index; as a first case suppose that \( p \) is a topological disk with two corners. Then \( p \) must have at least two switches, whence \( p \cap \partial S^b \) is a curve containing at least two negative rung conduit segments traversing upward ladders. By Lemma 6.3, \( p \cap \partial S^b \) is part of the boundary of a \( \partial \)-disk \( \delta \). Note that \( \partial \delta \) must have at least two positive rung conduit segments crossing upward ladders, forcing the corresponding nonmeridional disk patch of \( S^b \cap B^s \) to have at least four switches, contradicting the stable bigon property.

Suppose now that \( p \) is a topological annulus. Then \( p \cap \partial S^b \) is a closed curve in \( \partial N \) containing only negative conduits. A component of \( \partial S^b \) contains a negative rung conduit segment iff it is the boundary of a \( b \)-disk by Lemma 6.3, and the boundary of a \( b \)-disk must contain positive conduit segments. Hence \( p \cap \partial S^b \) must contain only ladderpole conduits, so \( p \) has no switches by Lemma 3.6.

The same analysis holds symmetrically for \( N \cap B^u \).

Lemma 7.4. Each component of \( N \) is an annulus.

Proof. We show that no component of \( N \) is a disk. It then follows from Lemma 7.3 that no component of \( N \) is an annulus.

Suppose for a contradiction that \( D \) is a disk component of \( N \). Let \( D^s = D \cap B^s \) and \( D^u = D \cap B^u \). Note that each patch of \( D^s \) or \( D^u \) is topologically a disk and has two corners, so has index equal to \( 1 - \# \{ \text{switches of the patch} \} \).
Within this proof we call a patch of $D^s$ or $D^u$ small if it intersects a single negative plate of $S^b$. If a patch is not small we will call it large. Let $p$ be a small patch of $D^s$. If $p$ has a switch, then $p \cap \partial M$ is a negative conduit in the boundary of a $b$-disk $\delta$. By the efficient bigon property $\partial \delta$ must have at least two negative conduits and they must be unlinked from the positive conduits in $\partial \delta$. This is a contradiction since $p$ is small. Hence $p$ must have no switches, and we conclude $\text{index}(p) = 1$. It follows that $D$ must contain more than one negative plate of $S^b$.

Note that each patch of $D^s$ can have at most one switch, for a patch of $D^s$ with more than one switch would imply that there is a $b$-disk of $S^b$ with two negative ladderpole conduit segments, contradicting the bigon property. Hence 0 and 1 are the only possible indices of patches of $D^s$. We can define a graph $G_D$ in $D$ with a vertex in the middle of each negative plate, and an edge connecting two vertices exactly when those two plates are connected by a negative rod. Since $D$ is a disk, $G_D$ is a connected tree, hence has at least two leaves (vertices of valence 1). Each leaf must lie interior to a negative plate which abuts two mixed rods and one negative rod. It follows that each of these plates intersects a small patch of $D^s$ and hence $D^s$ has at least two small patches. Since each small patch has index 1 by the above, and the sum of all the indices of $D^s$ is 2, $D^s$ must have exactly two small patches with index 1 and all other patches must have index 0. The same analysis holds for $D^u$. Hence there are two components of $D \cap \partial M$ consisting of a single conduit segment. Call these $\gamma_1$ and $\gamma_2$. There are two components of $(D \cap \text{neg}(S^b)) - (\gamma_1 \cup \gamma_2)$ (note that $D \cap \text{neg}(S^b)$ is just $D$ before rounding), which we call $A$ and $B$. The situation is as in Figure 27.

Let $\Delta_i$ be the plate of $D$ incident to $\gamma_i$ for $i = 1, 2$. Note that the switch of $\Delta_1 \cap B^s$ must point toward either $A$ or $B$ since the patch incident to $\gamma_1$ has index 1; suppose without loss of generality that it points toward $A$. This forces the switch of $\Delta_1 \cap B^u$ to point toward $B$. By Lemma 7.1, $S^b$ must curl up at $\Delta_1 \cap A$ and down at $\Delta_1 \cap B$.

Applying Lemma 7.2 repeatedly, we see that the curling of $S^b$ must be consistent at all rods meeting $A$ and at all rods meeting $B$. We conclude that no switches of $D^s$ point toward $B$ and in particular that the switch of $\Delta_2 \cap B^s$ points toward $A$, as in Figure 27.

It follows that we may delete all branches of $D^s$ which meet $B - (\Delta_1 \cap \Delta_2)$ and what remains will be another train track $(D^s)'$. By the above analysis there is a unique patch $p'$ of $(D^s)'$ meeting $B - (\Delta_1 \cap \Delta_2)$ and it has two corners and at least two switches, hence has index $\leq -1$. This forces some patch of $D^s$ to have negative index, a contradiction. We conclude that no component of $N$ is a disk. \hfill $\square$

**Lemma 7.5.** Each plate in $\text{neg}(S^b)$ is incident to a mixed rod.

**Proof.** Let $A$ be a component of $\text{neg}(S^b)$. By Lemma 7.4, $A$ is an annulus. Suppose there is a proper subset $Y$ of the set of plates in $A$ such that the union $A'$ of $Y$ with all rods
Figure 28. Two types of components of neg(S^5): ladderpole (left) and non-ladderpole (right). Note that each plate has an edge on \( \partial N \).

As a visual shorthand, we will draw pictures of these components omitting the rods in the plate and rod decomposition, simply drawing them as edges.

incident to two plates in \( Y \) is an annulus. Let \( A'' \) be the annulus obtained by rounding the corners of \( A' \). Let \( p'' \) be a patch of \( A'' \cap B^s \). Then \( p'' \) is contained in some patch \( p \) of \( A^s \) and \( \text{index}(p'') \geq \text{index}(p) \) since the number of corners is equal for both patches and \( p'' \) has at most as many switches as \( p \). The same holds for patches of \( A'' \cap B^u \), so we conclude that each patch of \( A'' \cap B^s \) and \( A'' \cap B^u \) has nonnegative index. Since \( \chi(A'') = 0 \), the index of each patch must be 0.

Because \( Y \) does not contain all the plates in \( A \) by assumption, there must be some negative rod \( r \) in \( A \) incident to a plate \( P \) in \( A - Y \). But because the index of any patch of \( A' \cap B^s \) or \( A' \cap B^u \) is 0, each patch touching \( r \) has a switch. Hence either \( (A' \cup r \cup P) \cap B^s \) or \( (A' \cup r \cup P) \cap B^u \) has a patch with two switches, contradicting Lemma 7.3.

It follows that there is no such set \( Y \) and the claim is proved. □

In light of Lemma 7.4 and Lemma 7.5, there are two types of possible components of \( \text{neg}(S^5) \). If \( A \) is a component of \( \text{neg}(S^5) \) such that some component of \( A \cap \partial M \) is a circle, we say \( A \) is a **ladderpole component of \( \text{neg}(S^5) \)**. Otherwise, \( A \) is a **non-ladderpole component of \( \text{neg}(S^5) \)** (see Figure 28). Ladderpole components are so named because if \( A \) is a ladderpole component and \( c \) is the component of \( \partial A \) lying in \( \partial N_c \), then \( c \) has ladderpole slope and consists of only ladderpole conduits by Lemma 7.3.

By Lemma 7.2 if \( A \) is a component of \( \text{neg}(S^5) \) and \( C \) is a component of \( \partial A \), then the curling of \( S^5 \) is consistent at each mixed rod incident to \( C \).

**Lemma 7.6.** Let \( A \) be a non-ladderpole component of \( \text{neg}(S^5) \). Let \( A^s := A \cap B^s \) and \( A^u := A \cap B^u \). If either \( A^s \) or \( A^u \) contains a large branch then the area of \( S^5 \) can be reduced by flat isotopy.

**Proof.** Suppose that \( A^s \) contains a large branch. Then after a finite sequence of upward flip moves on \( S^5 \), we can replace \( A \) by a new annulus \( A_+ \) with \( \partial A_+ = \partial A \) with the property that \( A_+ \) does not contain any large branches, and that \( A_+ \) has switches pointing toward \( \partial A_+ \). By Lemma 7.1, either \( S^5 \) curls up at \( \partial A \) or \( \partial S^5 \) has backtracking, in which case a face move reduces the area of \( S^5 \). Hence we can assume that \( S^5 \) curls up at \( \partial A \). Since \( A_+ \) was obtained by flipping \( A \) upwards, \( A_+ \) must have a large branch and we can flip \( A_+ \) downward to an annulus \( A_- \) with \( \partial A_- = \partial A = \partial A_+ \) such that \( A_- \) has no large branches and has switches pointing toward \( \partial A_- \). By Lemma 7.1, either \( S^5 \) curls down at \( \partial A \) or \( \partial S^5 \) has backtracking.
Since we assumed \( S^0 \) curls up at \( \partial A \), we conclude the area of \( S^0 \) can be reduced. A symmetric argument proves the claim for \( A^u \).

**Lemma 7.7.** Let \( A \) be a non-ladderpole component of \( \text{neg}(S^0) \). If neither \( A^s \) or \( A^u \) contains a large branch then one of \( A^s \) or \( A^u \) carries the core curve of \( A \), denoted \( \text{core}(A) \), and the image of \( \text{core}(A) \) under the collapsing map \( \text{coll}: N \rightarrow \tau(2) \) is a stable or unstable loop.

**Proof.** Suppose that \( A^s \) does not carry the core of \( A \). Note that the endpoints of any line segment \( \gamma \) properly embedded in \( A \) and carried by \( A^s \) must lie on different components of \( \partial A \). Otherwise, letting \( A' \) denote the component of \( N \) containing \( A \), one component of \( A' - \gamma \) would have index 1, forcing some patch of \( A' \cap B^s \) to have positive index.

To show that \( A^u \) carries the core of \( A \), we will show that \( A^u \) has no switches which point toward \( \partial A \). Let \( P \) be a plate in \( A \) incident to negative rods \( r_1 \) and \( r_2 \), and a mixed rod \( r_3 \). Let \( c_i \) be the \( \tau \)-edge corresponding to \( r_i \) for \( i = 1, 2, 3 \). For convenience we can choose the labeling \( r_1, r_2, r_3 \) to be oriented counterclockwise. Let \( p^u \) and \( p^s \) be the switches of \( A^u \) and \( A^s \) lying in \( P \). If \( p^s \) points toward \( \partial A \), then \( p^u \) must point toward \( r_1 \) or \( r_2 \) by **Lemma 3.4**. Now suppose that \( p^s \) points toward \( r_1 \) or \( r_2 \); we can assume without loss of generality that \( i = 2 \), which by our labeling forces \( r_1 \) to be right veering and \( r_3 \) to be left veering. Let \( Q \) be the plate in \( A \) which is also incident to \( r_2 \), and let \( q^u \) and \( q^s \) be the switches of \( A^u \) and \( A^s \) respectively lying in \( Q \). Label the other rods incident to \( Q \) by \( r_4 \) and \( r_5 \) so that the triple \((r_2, r_4, r_5)\) is oriented counterclockwise, as in the following picture:

![Diagram](image)

Suppose that \( r_2 \) is left veering. Then \( q^s \) must point toward \( r_5 \) since \( A^s \) has no large branches. If \( r_5 \) is a mixed rod then \( A^s \) must have a patch of index 0, a contradiction. If \( r_5 \) is negative then \( r_4 \) must be mixed, so since \( A^s \) does not carry the core of \( A \) there must be a properly embedded line segment starting at either \( r_3 \) or \( r_4 \) with both its endpoints on the same boundary component of \( A \), a contradiction. We conclude that \( r_2 \) must be right veering, forcing the large half branch incident to \( p^u \) to end at \( r_1 \) by **Lemma 3.4**. Since \( A^u \) has no switches pointing toward \( \partial A \), it carries \( \text{core}(A) \). Since \( A^u \) has no large branches, \( \text{core}(A) \) collapses to an unstable loop. Symmetric reasoning shows that if \( A^u \) does not carry the core of \( A \) then \( A^s \) does, and that the core then collapses to a stable loop. \( \square \)

If \( A \) is a non-ladderpole component and \( A^s \) carries \( \text{core}(A) \), which then necessarily collapses to a stable loop, then \( A \) is called a **stable component**. The definition of **unstable component** is symmetric. We remark that if \( A \) is a ladderpole component then \( \text{core}(A) \) collapses to a curve which is both a stable and unstable loop, and so it might be more precise to call stable and unstable components “strictly stable” and “strictly unstable,” but we forgo this precision.

**Definition 7.8.** Let \( S^0 \) be a flattened surface with the efficient bigon property. If in addition each component of \( \text{neg}(S^0) \) is either stable, unstable, or ladderpole, we say that \( S^0 \) has the **excellent bigon property**.
Remark 7.9. Suppose that \( S^q \) has the excellent bigon property. Because \( \partial \hat{S}^q \) has no backtracking, \( S^q \) curls down along stable components of \( \text{neg}(S^q) \) and up along the boundaries of unstable components. If \( A \) is a ladderpole component, \( S^q \) could possibly curl up or down along \( A \).

We compile the results of this subsection by recording the following useful lemma.

**Lemma 7.10.** Let \( S^q \) be a surface with the efficient bigon property. Then after a flat isotopy which does not increase area, \( S^q \) has the excellent bigon property.

7.3. Leveraging the excellent bigon property.

**Lemma 7.11.** Let \( S^q \) be a surface with the excellent bigon property, and let \( \delta \) be a \( \beta \)-disk of \( S^q \). Then \( \delta^- \) contains exactly two rung segments \( r_1 \) and \( r_2 \) which satisfy the following:

(i) \( r_1 \) and \( r_2 \) are adjacent in the sense that they are incident to the same negative junction segment, and

(ii) at least one of the \( r_i \) is incident to a cusp of \( \delta \).

**Proof.** The fact that \( \delta^- \) contains exactly two rung conduit segments follows from the efficient bigon property. Since \( S^q \) has the excellent bigon property, \( \delta^- \) is part of the boundary of a component \( A \) of \( N \) containing a stable or unstable loop and neither \( A^s = A \cap B^s \) nor \( A^u = A \cap B^u \) has a large branch. As such, \( \delta^- \) never traverses two conduits whose union corresponds to both \( 0-\pi \) edges of any flat triangle.

We shall prove (i) in the case when \( \delta \) is outward as the inward case is symmetric. Suppose that there is a segment of \( \delta^- \) whose conduit segments, in order, are \( r_1, \ell_1, \ldots, \ell_n, r_2 \) where \( r_1 \) crosses an upward ladder, \( r_2 \) crosses a downward ladder, and the \( \ell_i \) are ladderpole conduit segments. Let \( j \) be the junction of \( \ell_n \) and \( r_2 \), let \( s(j) \) be the corresponding \( \partial\tau^{(2)} \)-switch, and let \( b(\ell_n) \) and \( b(r_2) \) be the \( \partial\tau^{(2)} \)-branches corresponding to \( \ell_n \) and \( r_2 \) respectively. By the observation in the first paragraph of this proof, \( b(r_2) \) must not be the topmost branch at \( s(j) \). However, this means that \( b(r_2) \) must be a topmost branch at the switch corresponding to its other endpoint. Since \( \delta^- \) is the bottom of \( \delta \) we see that there must be another negative conduit segment \( \ell_{n+1} \) in \( \delta^- \) after \( r_2 \). If \( t \) is the flat triangle immediately above \( b(r_2) \), then \( r_2 \) and \( \ell_{n+1} \) correspond to both \( 0-\pi \) edges of \( t \), a contradiction. This proves (i).

To see the truth of (ii), we again assume \( \delta \) is outward and use the same notation as above, where \( r_1 \) crosses an upward ladder and \( r_2 \) crosses a downward ladder. Then \( r_1 \) must be incident to a cusp of \( \delta \) because otherwise there would be backtracking in \( \partial S^q \). Again, the inward case is symmetric. \( \square \)

If \( \delta \) is a \( \beta \)-disk of \( S^q \) with the excellent bigon property and \( \delta^- \) contains any ladderpole conduit segments in addition to the two adjacent rung conduit segments described in the previous lemma, we say \( \delta \) has **long negative boundary**. Otherwise \( \delta \) has **short negative boundary** (see Figure 29). We will orient \( \delta^- \) so that it starts at a rung conduit and ends at a ladderpole conduit. With this orientation, we call the first conduit of \( \delta^- \) the **initial conduit** of \( \delta^- \). All other conduits are called **noninitial conduits**. The first negative junction segment traversed by \( \delta^- \) is called its initial junction and all other negative conduit segments are called **noninitial junctions**. Note that \( \delta^- \) has noninitial junctions if and only if it is long.

**Lemma 7.12** (noninitial rung lemma). Let \( S^q \) be a surface with the excellent bigon property. Let \( \delta \) be a \( \beta \)-disk of \( S^q \) with long negative boundary, which we think of as a concatenation of conduits \( r_1, r_2, \ell_1, \ldots, \ell_n \) where the \( r_i \) are rungs and the \( \ell_i \) are ladderpoles. If the flat triangle \( t \) meeting \( r_2 \) on the \( \delta \) side is non-hinge, then the area of \( S^q \) can be reduced by flat isotopy.
Figure 29. An efficient $b$-disk $\delta$. If $\delta$ is outward, then the green segment is the negative boundary and it is oriented as shown. If $\delta$ is inward, the magenta segment is the negative boundary and it is oriented as shown. In both cases the negative boundary is long.

Proof. We prove the lemma only in the case when $\delta$ is outward, as the proof in the inward case is symmetric. In this case $\delta^-$ is the bottom of $\delta$, so that $t$ is the flat triangle immediately above $r_2$. The fact that $\delta^-$ is the bottom of $\delta$ also tells us that if $A$ is the component of $N$ corresponding to $\delta$, then $S^5$ must curl down at each mixed rod incident to $A$.

Let $e$ be the $\tau$-edge which meets $\partial M$ at the junction of $r_2$ with $\ell_1$, and let $\otimes$ be the tetrahedron corresponding to $t$. Let $t'$ be the tip of $\otimes$ on the other side of $e$ from $t$, so that $e$ meets $t'$ at one of its 0-vertices $v$. Note that $t'$ is non-hinge and the fan on the $t'$-side of $v$ is long. If $v'$ is the other 0-vertex of $t'$, the fan on the $t'$-side of $v'$ is short by Lemma 3.2. Since $S^5$ curls down along $\partial A$, there is a tetrahedron move of $S^5$ across $\otimes$. If $\delta$ is not innermost then this move is possibly of a $b$-disk contained in $\delta$. Either way, the move creates a kink in the boundary of some $b$-disk. For pictures see Figure 30.

Now by Lemma 6.6 and Lemma 6.9, we can perform a flat isotopy to reduce the area of $S^5$. □

Lemma 7.13 (nonterminal ladderpole lemma). Let $S^5, \delta, r_1,r_2, \ell_1, \ldots, \ell_n$ be as in Lemma 7.12 and suppose $n \geq 2$. If the flat triangle $t$ on the $\delta$-side of $\ell_i$ for $i = 1, \ldots, n - 1$ is hinge, then the area of $S^5$ can be reduced by flat isotopy.
Figure 31. Pictures in the proof of Lemma 7.13, left to right: if $t$ is hinge, there is an available tetrahedron move across $\otimes$, after which $S^\circ$ no longer has the unstable bigon property.

Proof. Again, we will prove the lemma only in the case when $\delta$ is outward, in which case $\delta^-$ is the bottom of $\delta$. Let $\otimes$ be the tetrahedron corresponding to $\delta$. Suppose for a contradiction that $t$ is hinge. Let $u$ be the $\tau$-edge terminating at $u$, let $v$ be the other endpoint of $e$ on $\partial M$, and let $t'$ be the tip of $\otimes$ on the other side of $e$ from $u$, so that $v$ is a $0$-vertex of $t'$. Let $v'$ be the other $0$-vertex of $t'$. Note that $t'$ is an upward flat triangle and that it is not topmost at $v$, so it is topmost at $v'$. Since $S^\circ$ curls down along $\partial A$, there is a tetrahedron move of $S^\circ$ across $\otimes$. As in Lemma 7.12, this is true regardless of whether $\delta$ is innermost. After this tetrahedron move $S^\circ$ does not have the bigon property. By Lemma 6.6 we can reduce the area of $S^\circ$ by flat isotopy. □

Lemma 7.14 (annulus lemma). Let $S^\circ$ be a surface with the excellent bigon property. Let $A$ be a stable component of $\partial neg(S^\circ)$. Then core($A$) collapses to a shallow stable loop. Symmetrically, if $A$ is unstable then core($A$) collapses to a shallow unstable loop.

Hence if $\partial neg(S^\circ)$ has a non-ladderpole component, there is an available annulus move $S^\circ$ after which $\partial neg(S^\circ)$ has ladderpole components.

Proof. It follows from Lemma 7.12 and Lemma 7.13 that if $\delta$ is an outward $b$-disk with long negative boundary, then at each noninitial junction, $\delta^-$ passes from a second-topmost to a topmost conduit. Symmetrically, if $\delta$ is an inward $b$-disk with long negative boundary then at each junction of noninitial segments, $\delta^-$ passes from a second-bottommost to a bottommost conduit.

Similarly to the preceding lemmas, the case when $A$ is stable is illustrative; suppose $A$ is stable. Each patch of $A^\circ$ intersects $\partial M$ in the negative boundary of a $b$-disk. Then for each negative rod $r$ in $A$, there is a component of $r \cap \partial M$ which is noninitial junction (see Figure 32). It follows that the core of $A$ collapses to a shallow stable loop.

If $A$ is topmost in each of its $N_r$-plates, we may perform an annulus move and the proof is complete. Otherwise after a finite (possibly empty) collection of upward flips, we may perform an annulus move. □

The following lemma is the final piece we need to prove Theorems A and B from the introduction.
Figure 32. A stable component of neg(S^0). We have indicated long negative boundaries by green segments, and noninitial junctions by dots.

Figure 33. Pictures from the proof of Lemma 7.15.

Lemma 7.15 (ladderpole component lemma). Let S^0 be a surface with the excellent bigon property. Suppose neg(S^0) has a ladderpole component A. Then the area of S^0 can be reduced by flat isotopy.

Proof. Suppose that S^0 curls down at \partial A; the argument when S^0 curls up is symmetric. There is a component U_A of U such that \partial A meets \partial U_A in a curve \lambda containing only ladderpole conduit segments. Then there is a hinge downward flat triangle t which is incident to a conduit segment c of \lambda. Let \bigotimes be the tetrahedron corresponding to t, and let u be the endpoint of c which is higher on \lambda. Let t', v, and v' be as in Figure 33. Then t is bottommost in its side of u, and hence t' is bottommost in its side of v and topmost in its side of v' by Lemma 3.2. Since S^0 curls down along \partial A, there must be an available tetrahedron move across \bigotimes. The effect of this move on \lambda shows that S^0 no longer has the bigon property. By Lemma 6.6, S^0 is flat isotopic to a surface with less area. □

Theorem 7.16. Let \tau be a veering triangulation of \hat{M}, and let M be obtained from M by Dehn filling along slopes with \geq 3 prongs. Let S be a taut surface such that [S] \in \text{cone}(\sigma_\tau). Then S is carried by \tau^{(2)} up to isotopy.

Proof. Let S^0 be a flattening of S that minimizes area in its flat isotopy class. By Lemma 6.9, S^0 has the efficient bigon property. We wish to show that neg(S^0) is empty. By Lemma 7.10,
$S^9$ has the excellent bigon property. By Lemma 7.14, if neg($S^9$) is nonempty then up to a flat isotopy which does not increase area we can assume neg($S^9$) has ladderpole components. However, if neg($S^9$) had a ladderpole component then $S^9$ would not minimize area by Lemma 7.15. We conclude that neg($S^9$) = $\emptyset$, so $S^9$ is carried by $\tau^{(2)}$. □

The above proof is nonconstructive as written, but within it is a recipe for producing the carried surface. We provide this recipe explicitly now:

(0) Given a taut surface $S$ with $[S] \in \text{cone}(\sigma_\tau)$, flatten it and call the resulting surface $S^9$. Proceed to step 1.

(1) Is neg($S^9$) empty?
   (a) If so, then $S^9$ is carried by $\tau^{(2)}$ and we are done.
   (b) If not, proceed to step 2.

(2) Does $\partial S^9$ have backtracking?
   (a) If so, reduce the area of $S^9$ by a face move and return to step 1.
   (b) If not, proceed to step 3.

(3) Does $S^9$ have a $b$-disk of width $\leq 1$?
   (a) If so, reduce the area of $S^9$ by a width one move and return to step 1.
   (b) If not, then $S^9$ has the bigon property by the proof of Lemma 6.2. Proceed to step 4.

(4) Does $S^9$ have the efficient bigon property?
   (a) If not, reduce the area of $S^9$ using methods from the proof of Lemma 6.9 and return to step 1.
   (b) If so, proceed to step 5.

(5) Does $S^9$ have the excellent bigon property?
   (a) If not, reduce the area of $S^9$ using methods from the proof of Lemma 7.6 and return to step 1.
   (b) If so, proceed to step 6.

(6) Does neg($S^9$) have a ladderpole component?
   (a) If so, reduce the area of $S^9$ using methods from the proof of Lemma 7.15 and return to step 1.
   (b) If not, then the core of every component of neg($S^9$) collapses to a shallow stable or unstable loop. Perform an annulus move, after which neg($S^9$) has a ladderpole component. Reduce the area of $S^9$ by Lemma 7.15 and return to step 1.

Theorem 7.16 immediately gives the following:

Corollary 7.17. cone($\sigma_\tau$) $\subset C_{C_\tau}$, so cone($\sigma_\tau$) = $C_{C_\tau}$.

Proof. If $\alpha$ is an integral class in cone($\sigma_\tau$) then Theorem 7.16 produces a surface representing $\alpha$ carried by $\tau^{(2)}$, which clearly pairs nonnegatively with each closed positive transversal to $\tau^{(2)}$ lying in $M$. □

As a summary of the paper to this point we provide a proof of Theorem A and Theorem B.

Theorems A and B. Let $\tau$ be a veering triangulation of a compact 3-manifold $\tilde{M}$. If $M$ is obtained by Dehn filling each component of $\partial \tilde{M}$ along slopes with $\geq 3$ prongs then $M$ is irreducible and atoroidal. Let $\sigma_\tau$ be the face of the Thurston norm ball $B_x(M)$ determined by the Euler class $e_\tau$. Then the following hold:

(i) cone($\sigma_\tau$) = $C_{C_\tau}$, and the codimension of $\sigma_\tau$ in $\partial B_x(M)$ is equal to the dimension of the largest linear subspace contained in $C_\tau$. 

(ii) If $S \subset M$ is a surface, then $S$ is carried by $\tau^{(2)}$ up to isotopy if and only if $S$ is taut and $[S] \in \text{cone}(\sigma_\tau)$.

Proof. We get irreducibility of $M$ from Lemma 4.1 and atoroidality by Lemma 5.2.

The containment $C_\tau^+ \subset \text{cone}(\sigma_\tau)$ is given by Corollary 5.6. The reverse containment is given by Corollary 7.17. Now the claim about the dimension of $\sigma_\tau$ in statement (i) is basic linear algebra and follows from the fact that $C_\tau$ and $C_\tau^+$ are dual convex polyhedral cones, see e.g. [Ful93, §1.2]. This completes the proof of (i).

If $S$ is carried by $\tau^{(2)}$ up to isotopy then $S$ is taut and $[S] \in \text{cone}(\sigma_\tau)$ by Lemma 5.4. The other direction of statement (ii) is Theorem 7.16. \qed

8. The case with boundary

Now we explain how to modify our methods to obtain a result in the case when the manifold we care about is the one triangulated by $\tau$, rather than a Dehn filling. The result in this section is used in [LMT20] to show that a veering triangulation determines a face of the Thurston norm ball of $H_2(M, \partial M)$.

We begin by expanding our definition of partial branched surface. Let $M$ be a compact 3-manifold with boundary. A partial branched surface in $M$ is a branched surface $B$ in $M - \text{int}(U)$, where $U$ is a union of closed solid tori and closed regular neighborhoods of components of $\partial M$. A properly embedded surface $S \subset M$ is carried by a partial branched surface $B \subset M$ if $S$ has no components completely contained in $U$, $S - \text{int}(U)$ is carried by $B$, and each component $S \cap U$ $\pi_1$-injects into its component of $U$.

Let $\tau$ be a veering triangulation of a compact manifold $M$. Let $M$ be obtained by gluing a thickened torus $T^2 \times [0, 1]$ to each boundary component of $M$. Then $\tau^{(2)}$ is a partial branched surface in $M$ with $U = M - \tilde{M}$.

We define $N_\varepsilon$ just as in Section 4.2. A flattened surface $S^\circ$ in $M$ is a properly embedded incompressible and $\partial$-incompressible surface in $M$ such that $S^\circ \cap \tilde{M}$ lies in $N_\varepsilon$ transverse to the vertical foliation and each component of $S^\circ \cap U$ is $\pi_1$-injective in its component of $U$. Terms like plate, rod, and area translate immediately to this setting. A $b$-disk of $S^\circ$ is a disk in $\partial \tilde{M}$ bounded by a component of $S^\circ \cap \partial \tilde{M}$. Note that if $\delta$ is a $b$-disk, then $\partial \delta$ must also bound a disk in $S^\circ \cap U$. Words like inward, outward, volume, and cusp that we previously used to describe $b$-disks in the case of a closed 3-manifold also translate immediately.

Theorem 8.1. Let $S$ be an incompressible and boundary incompressible surface in $M$, where $M$ is obtained by gluing a thickened torus to each component of $\tilde{M}$. Further suppose that $S$ has the property that for any surface $S'$ isotopic to $S$ that is transverse to $B^u$ and $B^s$, either:

- one of $S' \cap B^u$ or $S' \cap B^s$ has a patch of positive index, or
- every negative switch of $S' \cap B^u$ and every negative switch of $S' \cap B^s$ belongs to a bigon.

Then $S$ is isotopic to a surface carried by the partial branched surface $\tau^{(2)} \cap \tilde{M}$.

Proof. We first flatten $S$ to a flattened surface $S^\circ$. This works exactly as in Section 4.3. Now our definitions of flat isotopies translate verbatim to this setting. Recall that the face move in the closed case used irreducibility of the closed manifold. As noted in the proof of Lemma 5.2 $M$ is both irreducible and atoroidal, so there is no problem. Hence we can apply face moves and width 1 moves until $S^\circ$ has no $b$-disks of width 1, and thus neither $S^\circ \cap B^u$ nor $S^\circ \cap B^s$ has a patch of positive index. By the condition in the theorem statement, every negative switch of $S^\circ \cap B^s$ belongs to a bigon and similarly for $B^u$. We can now apply our argument
from the closed case to reduce the area of $S^9$ until it has no negative plates, beginning with Lemma 6.3 and proceeding directly through the proof of Theorem 7.16. \hfill \square

References


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