Math 132: Discussion Session: Week 10

Directions: In groups of 3-4 students, work the problems on the following page. Below, list the members of your group and your answers to the specified questions. Turn **this paper** in at the end of class. You do not need to turn in the question page or your work.

Additional Instructions: It is okay if you do not completely finish all of the problems. Also, each group member should work through each problem, as similar problems may appear on the exam.

Scoring:

Correct answers	Grade
0–2	0%
3–5	80%
6–8	100%

Group Members:

11.2: Series.

(1) (a) $\displaystyle\sum_{n=1}^{\infty} \frac{2 \cdot 3^n}{(-4)^n} =$

(b) $\displaystyle\sum_{n=1}^{\infty} \left(e^{1/n} - e^{1/(n+1)}\right) =$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{6 \cdot 2^{2n-1}}{3^n} =$

(d) $\displaystyle\sum_{n=1}^{\infty} (\sin 100)^n =$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{n+2}{2n+5} =$

(f) $\displaystyle\sum_{n=2}^{\infty} \frac{3^n + n}{2^n + 4} =$

(2) (a) $\displaystyle\sum_{n=1}^{\infty} (-5)^n x^n =$

- converges whenever x is strictly between $\boxed{}$ and $\boxed{}$.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n}$

- converges whenever x is strictly between $\boxed{}$ and $\boxed{}$.

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11.2: Series.

(1) Determine whether the series converges or diverges. If it converges, determine its value.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2 \cdot 3^n}{(-4)^n}$

Solution: This looks like a geometric series. Expanding the summation, we see that

$$\sum_{n=1}^{\infty} \frac{2 \cdot 3^n}{(-4)^n} = \frac{2 \cdot 3}{-4} + \frac{2 \cdot 3^2}{(-4)^2} + \frac{2 \cdot 3^3}{(-4)^3} + \frac{2 \cdot 3^4}{(-4)^4} + \cdots$$

Factoring out the first term, we see that

$$\sum_{n=1}^{\infty} \frac{2 \cdot 3^n}{(-4)^n} = \frac{2 \cdot 3}{-4}\left(1 + \left(-\frac{3}{4}\right) + \left(-\frac{3}{4}\right)^2 + \left(-\frac{3}{4}\right)^3 + \cdots\right)$$

Since, $\left|-\frac{3}{4}\right| < 1$, we can sum the geometric series, so

$$\sum_{n=1}^{\infty} \frac{2 \cdot 3^n}{(-4)^n} = \frac{2 \cdot 3}{-4}\left(\frac{1}{1 - \left(-\frac{3}{4}\right)}\right) = \frac{2 \cdot 3}{-4}\left(\frac{4}{7}\right) = \boxed{-\frac{6}{7}}.$$

(b) $\displaystyle\sum_{n=1}^{\infty} \left(e^{1/n} - e^{1/(n+1)}\right)$

Solution: This looks like it might be a telescoping series. Expanding out the summation, we see that

$$\sum_{n=1}^{\infty} \left(e^{1/n} - e^{1/(n+1)}\right) = \left(e^1 - e^{1/2}\right) + \left(e^{1/2} - e^{1/3}\right) + \left(e^{1/3} - e^{1/4}\right) + \left(e^{1/4} - e^{1/5}\right) + \cdots$$

Indeed, there's going to be cancellation when we compute the partial sums. We compute that

$$\begin{aligned} s_1 &= e^1 - e^{1/2}, \\ s_2 &= e^1 - e^{1/3}, \\ s_3 &= e^1 - e^{1/4}, \\ s_4 &= e^1 - e^{1/5}. \end{aligned}$$

In general $s_n = e - e^{1/(n+1)}$. Our next task is to compute $\lim_{n\to\infty} s_n$. As n becomes very large, $1/(n+1)$ becomes very close to 0. Since the exponential function is continuous, we see that

$$\lim_{n\to\infty} \left(e - e^{1/(n+1)}\right) = e - e^0 = \boxed{e - 1}.$$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{6 \cdot 2^{2n-1}}{3^n}$

Solution: This looks like a geometric series. Expanding the summation, we see that

$$\sum_{n=1}^{\infty} \frac{6 \cdot 2^{2n-1}}{3^n} = \frac{6 \cdot 2^1}{3^1} + \frac{6 \cdot 2^3}{3^2} + \frac{6 \cdot 2^5}{3^3} + \frac{6 \cdot 2^7}{3^4} + \cdots.$$

Factoring out the first term, we see that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{6 \cdot 2^{2n-1}}{3^n} &= \frac{6 \cdot 2^1}{3^1}\left(1 + \frac{\cdot 2^2}{3} + \frac{2^4}{3^2} + \frac{2^6}{3^3} + \cdots\right) \\ &= 4\left(1 + \frac{4}{3} + \left(\frac{4}{3}\right)^2 + \left(\frac{4}{3}\right)^3 + \cdots\right). \end{aligned}$$

Since $\left|\frac{4}{3}\right| > 1$, this geometric series $\boxed{\text{diverges}}$.

(d) $\displaystyle\sum_{n=1}^{\infty}(\sin 100)^n$

Solution: The key idea here is to not get scared of the $\sin 100$. It's just a number, about -0.506, though the exact value doesn't matter. Something like $\sum_{n=1}^{\infty}(-0.506)^n$ is clearly a geometric series. Expanding out the summation, we see that

$$\begin{aligned}\sum_{n=1}^{\infty}(\sin 100)^n &= \sin 100 + (\sin 100)^2 + (\sin 100)^3 + (\sin 100)^4 + \cdots \\ &= \sin 100\left(1 + \sin 100 + (\sin 100)^2 + (\sin 100)^3 + \cdots\right).\end{aligned}$$

We don't need to know the exact value of $\sin 100$, but we do need to know that $|\sin 100| < 1$ or else the series will diverge. Fortunately, we know that $|\sin x| \leq 1$. But the series would still diverge if $\sin 100$ were equal to 1 or -1. Is that possible? No. We know that $\sin x = \pm 1$ when $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$. The number 100 is not on this list. Thus, indeed, $|\sin 100|$ is *strictly* smaller than 1, so the series converges to

$$\sum_{n=1}^{\infty}(\sin 100)^n = \sin 100\left(\frac{1}{1-\sin 100}\right) = \boxed{\frac{\sin 100}{1-\sin 100}} = \boxed{\frac{1}{1-\sin 100} - 1}.$$

(e) $\displaystyle\sum_{n=1}^{\infty}\frac{n+2}{2n+5}$

Solution: To understand the sequence $\frac{n+2}{2n+5}$, we look at the leading terms, $\frac{n}{2n} = \frac{1}{2}$, so we see that $\lim_{n\to\infty}\frac{n+2}{2n+5} = \frac{1}{2}$. If we add up a ton of numbers that are all very close to $\frac{1}{2}$, we'll get a very big result. Thus, the partial sums get bigger and bigger, so the series $\boxed{\text{diverges}}$.

(f) $\displaystyle\sum_{n=2}^{\infty}\frac{3^n+n}{2^n+4}$

Solution: Each term of the sequence $\frac{3^n+n}{2^n+4}$ is bigger than one. If we add up a ton of terms that are bigger than one, we'll get a huge result. The partial sums get bigger and bigger, so the series $\boxed{\text{diverges}}$.

(2) Find the values of x for which the series converges. Determine the value of the series for those values of x.

(a) $\displaystyle\sum_{n=1}^{\infty}(-5)^n x^n$

Solution: The key here is to not get scared of variables. This is a geometric series. Expanding out the sum, we see that

$$\begin{aligned}\sum_{n=1}^{\infty}(-5)^n x^n &= -5x + (-5)^2x^2 + (-5)^3x^3 + (-5)^4x^4 + \cdots \\ &= (-5x)\left(1 + (-5x) + (-5x)^2 + (-5x)^3 + \cdots\right).\end{aligned}$$

This geometric series converges as long as $|-5x| < 1$, which happens as long as $|x| < \frac{1}{5}$, that is $\boxed{-\frac{1}{5} < |x| < \frac{1}{5}}$. When that is the case, the series converges to

$$\sum_{n=1}^{\infty}(-5)^n x^n = (-5x)\frac{1}{1-(-5x)} = \boxed{-\frac{5x}{1+5x}}.$$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n}$

Solution: This is also a geometric series. Keep in mind that the summation starts at $n = 0$, which makes the problem easier. Expanding out the sum, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n} &= 1 + \frac{x-2}{3} + \frac{(x-2)^2}{3^2} + \frac{(x-2)^3}{3^3} + \cdots \\ &= 1 + \left(\frac{x-2}{3}\right) + \left(\frac{x-2}{3}\right)^2 + \left(\frac{x-2}{3}\right)^3 + \cdots .\end{aligned}$$

This geometric series converges as long as $\left|\frac{x-2}{3}\right| < 1$, which can be simplified to $|x-2| < 3$, which means $\boxed{-1 < x < 5}$.