Math 132: Discussion Session: Week 10

Directions: In groups of 3-4 students, work the problems on the following page. Below, list the members of your group and your answers to the specified questions. Turn this paper in at the end of class. You do not need to turn in the question page or your work.

Additional Instructions: It is okay if you do not completely finish all of the problems. Also, each group member should work through each problem, as similar problems may appear on the exam.

Scoring:

<table>
<thead>
<tr>
<th>Correct answers</th>
<th>Grade</th>
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<tbody>
<tr>
<td>0–2</td>
<td>0%</td>
</tr>
<tr>
<td>3–5</td>
<td>80%</td>
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<tr>
<td>6–8</td>
<td>100%</td>
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Group Members:

11.2: Series.

(1) (a) \[ \sum_{n=1}^{\infty} \frac{2 \cdot 3^n}{(-4)^n} = \]

(b) \[ \sum_{n=1}^{\infty} \left( e^{1/n} - e^{1/(n+1)} \right) = \]

(c) \[ \sum_{n=1}^{\infty} \frac{6 \cdot 2^{n-1}}{3^n} = \]

(d) \[ \sum_{n=1}^{\infty} (\sin 100)^n = \]

(e) \[ \sum_{n=1}^{\infty} \frac{n + 2}{2n + 5} = \]

(f) \[ \sum_{n=2}^{\infty} \frac{3^n + n}{2^n + 4} = \]

(2) (a) \[ \sum_{n=1}^{\infty} (-5)^n x^n = \]

• converges whenever \( x \) is strictly between \( \square \) and \( \square \).

(b) \[ \sum_{n=0}^{\infty} \frac{(x - 2)^n}{3^n} = \]

• converges whenever \( x \) is strictly between \( \square \) and \( \square \).
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11.2: Series.

(1) Determine whether the series converges or diverges. If it converges, determine its value.

(a) $$\sum_{n=1}^{\infty} \frac{2 \cdot 3^n}{(-4)^n}$$

Solution: This looks like a geometric series. Expanding the summation, we see that

$$\sum_{n=1}^{\infty} \frac{2 \cdot 3^n}{(-4)^n} = \frac{2 \cdot 3}{-4} + \frac{2 \cdot 3^2}{(-4)^2} + \frac{2 \cdot 3^3}{(-4)^3} + \frac{2 \cdot 3^4}{(-4)^4} + \cdots$$

Factoring out the first term, we see that

$$\sum_{n=1}^{\infty} \frac{2 \cdot 3^n}{(-4)^n} = 2 \cdot 3 \left( \frac{1}{1 - (-\frac{3}{4})} \right) = \frac{2 \cdot 3}{-4} \left( \frac{4}{7} \right) = \frac{-6}{7}.$$ 

(b) $$\sum_{n=1}^{\infty} \left( e^{1/n} - e^{1/(n+1)} \right)$$

Solution: This looks like it might be a telescoping series. Expanding out the summation, we see that

$$\sum_{n=1}^{\infty} \left( e^{1/n} - e^{1/(n+1)} \right) = \left( e^1 - e^{1/2} \right) + \left( e^{1/2} - e^{1/3} \right) + \left( e^{1/3} - e^{1/4} \right) + \left( e^{1/4} - e^{1/5} \right) + \cdots$$

Indeed, there’s going to be cancellation when we compute the partial sums. We compute that

$$s_1 = e^1 - e^{1/2},$$
$$s_2 = e^1 - e^{1/3},$$
$$s_3 = e^1 - e^{1/4},$$
$$s_4 = e^1 - e^{1/5}.$$ 

In general $$s_n = e - e^{1/(n+1)}.$$ Our next task is to compute $$\lim_{n \to \infty} s_n.$$ As $$n$$ becomes very large, $$1/(n+1)$$ becomes very close to 0. Since the exponential function is continuous, we see that

$$\lim_{n \to \infty} \left( e - e^{1/(n+1)} \right) = e - e^0 = e - 1.$$ 

(c) $$\sum_{n=1}^{\infty} \frac{6 \cdot 2^{2n-1}}{3^n}$$

Solution: This looks like a geometric series. Expanding the summation, we see that

$$\sum_{n=1}^{\infty} \frac{6 \cdot 2^{2n-1}}{3^n} = \frac{6 \cdot 2^1}{3^1} + \frac{6 \cdot 2^3}{3^2} + \frac{6 \cdot 2^5}{3^3} + \frac{6 \cdot 2^7}{3^4} + \cdots.$$ 

Factoring out the first term, we see that

$$\sum_{n=1}^{\infty} \frac{6 \cdot 2^{2n-1}}{3^n} = 6 \cdot \frac{2^1}{3^1} \left( 1 + \frac{2^2}{3^2} + \frac{2^4}{3^3} + \frac{2^6}{3^4} + \cdots \right)$$

$$= 4 \left( 1 + \frac{4}{3} + \left( \frac{4}{3} \right)^2 + \left( \frac{4}{3} \right)^3 + \cdots \right).$$
Since $\left|\frac{4}{3}\right| > 1$, this geometric series diverges.

\[ \sum_{n=1}^{\infty} (\sin 100)^n \]

**Solution:** The key idea here is to not get scared of the $\sin 100$. It’s just a number, about $-0.506$, though the exact value doesn’t matter. Something like $\sum_{n=1}^{\infty} (-0.506)^n$ is clearly a geometric series. Expanding out the summation, we see that

\[
\sum_{n=1}^{\infty} (\sin 100)^n = \sin 100 + (\sin 100)^2 + (\sin 100)^3 + (\sin 100)^4 + \cdots \\
= \sin 100 \left(1 + (\sin 100)^2 + (\sin 100)^3 + \cdots\right).
\]

We don’t need to know the exact value of $\sin 100$, but we do need to know that $|\sin 100| < 1$ or else the series will diverge. Fortunately, we know that $|\sin x| \leq 1$. But the series would still diverge if $\sin 100$ were equal to $1$ or $-1$. Is that possible? No. We know that $\sin x = \pm 1$ when $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$. The number $100$ is not on this list. Thus, indeed, $|\sin 100|$ is strictly smaller than $1$, so the series converges to

\[
\sum_{n=1}^{\infty} (\sin 100)^n = \sin 100 \frac{1}{1 - \sin 100} = \frac{\sin 100}{1 - \sin 100} - 1.
\]

\[ \sum_{n=1}^{\infty} \frac{n+2}{2n+5} \]

**Solution:** To understand the sequence $\frac{n+2}{2n+5}$, we look at the leading terms, $\frac{n}{2n} = \frac{1}{2}$, so we see that $\lim_{n\to\infty} \frac{n+2}{2n+5} = \frac{1}{2}$. If we add up a ton of numbers that are all very close to $\frac{1}{2}$, we’ll get a very big result. Thus, the partial sums get bigger and bigger, so the series diverges.

\[ \sum_{n=2}^{\infty} \frac{3^n + n}{2^n + 4} \]

**Solution:** Each term of the sequence $\frac{3^n + n}{2^n + 4}$ is bigger than one. If we add up a ton of terms that are bigger than one, we’ll get a huge result. The partial sums get bigger and bigger, so the series diverges.

(2) Find the values of $x$ for which the series converges. Determine the value of the series for those values of $x$.

\[ \sum_{n=1}^{\infty} (-5)^n x^n \]

**Solution:** The key here is to not get scared of variables. This is a geometric series. Expanding out the sum, we see that

\[
\sum_{n=1}^{\infty} (-5)^n x^n = -5x + (-5)^2 x^2 + (-5)^3 x^3 + (-5)^4 x^4 + \cdots \\
= (-5x) \left(1 + (-5x) + (-5x)^2 + (-5x)^3 + \cdots\right).
\]

This geometric series converges as long as $|-5x| < 1$, which happens as long as $|x| < \frac{1}{5}$, that is $-\frac{1}{5} < x < \frac{1}{5}$. When that is the case, the series converges to

\[
\sum_{n=1}^{\infty} (-5)^n x^n = (-5x) \frac{1}{1 - (-5x)} = \frac{-5x}{1 + 5x}.
\]
Solution: This is also a geometric series. Keep in mind that the summation starts at \( n = 0 \), which makes the problem easier. Expanding out the sum, we have

\[
\sum_{n=0}^{\infty} \frac{(x - 2)^n}{3^n} = 1 + \frac{x - 2}{3} + \frac{(x - 2)^2}{3^2} + \frac{(x - 2)^3}{3^3} + \cdots
\]

This geometric series converges as long as \( \left| \frac{x - 2}{3} \right| < 1 \), which can be simplified to \( |x - 2| < 3 \), which means \(-1 < x < 5\).