Math 132: Discussion Session: Week 13

Directions: In groups of 3-4 students, work the problems on the following page. Below, list the members of your group and your answers to the specified questions. Turn **this paper** in at the end of class. You do not need to turn in the question page or your work.

Additional Instructions: It is okay if you do not completely finish all of the problems. Also, each group member should work through each problem, as similar problems may appear on the exam.

Scoring:

Correct answers	Grade
0–3	0%
4–6	80%
7–24	100%

Group Members:

11.8: Power Series.

(1) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^n\sqrt{n^2+1}}$ converges when _____ $< x <$ _____.

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3n}(x+3)^n$ converges when _____ $< x <$ _____.

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{(-5)^n}{n!}(x+10)^n$ converges when _____ $< x <$ _____.

(4) $\displaystyle\sum_{n=1}^{\infty} e^n(x-2)^n$ converges when _____ $< x <$ _____.

11.9: Representing Functions as Power Series.

(1) (a) $f(x) =$

Radius of convergence?

(b) $f(x) =$

Radius of convergence?

(c) $f(x) =$

Radius of convergence?

(d) $f(x) =$

Radius of convergence?

(2) (a) $\displaystyle\int \frac{t}{1-t^8}\, dt =$

Radius of convergence?

(b) $\displaystyle\int x^2 \ln(1+x)\, dx =$

Radius of convergence?

11.10: Taylor and Maclaurin series.

(1) (a) $f(x) \approx$

(b) $f(x) \approx$

(c) $f(x) \approx$

(d) $f(x) \approx$

(2) (a) $f(x) =$

(b) $f(x) =$

(c) $f(x) =$

(d) $f(x) =$

(3) (a) $f(x) =$

(b) $f(x) =$

Interval of convergence?

(c) $f(x) =$

(d) $f(x) =$

(4) (a) $\displaystyle\int \sin\left(x^3\right)\, dx$

(b) $\displaystyle\int \frac{\cos x - 1}{x}\, dx$

Math 132 Discussion Session: Week 13

11.8: Power Series. Find the range of values of x for which the following series converge. In your answer, change the $<$ symbol to a $\leq$ symbol when needed. If the range of values doesn't have a lower or upper bound, fill in the blanks with $-\infty$ and ∞.

(1) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^n\sqrt{n^2+1}}$

Solution: To use the ratio test, we compute that

$$a_n = \frac{(-1)^n x^n}{2^n\sqrt{n^2+1}}, \qquad a_{n+1} = \frac{(-1)^{n+1} x^{n+1}}{2^{n+1}\sqrt{(n+1)^2+1}}.$$

Then, we compute that

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{\frac{(-1)^{n+1}x^{n+1}}{2^{n+1}\sqrt{(n+1)^2+1}}}{\frac{(-1)^n x^n}{2^n\sqrt{n^2+1}}}\right| \\ &= \left|\frac{x^{n+1}}{x^n}\cdot\frac{2^n}{2^{n+1}}\cdot\frac{\sqrt{n^2+1}}{\sqrt{(n+1)^2+1}}\right| \\ &= |x|\cdot\frac{1}{2}\cdot\sqrt{\frac{n^2+1}{(n+1)^2+1}}\end{aligned}$$

To take the limit, we note that the leading term in the numerator is n^2, and the leading term in the denominator is also n^2, so

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}|x|\cdot\frac{1}{2}\cdot\sqrt{\frac{n^2}{n^2}} = |x|\cdot\frac{1}{2}\cdot 1 = \frac{|x|}{2}.$$

The ratio test tells us that the series converges when $\frac{|x|}{2} < 1$ and diverges when $\frac{|x|}{2} > 1$. When $\frac{|x|}{2} = 1$, we need to investigate further. In other words, we know that the series converges when $-2 < x < 2$. When $x = \pm 2$, we need to investigate further.

When $x = -2$, the series becomes

$$\sum_{n=0}^{\infty}\frac{(-1)^n(-2)^n}{2^n\sqrt{n^2+1}} = \sum_{n=0}^{\infty}\frac{1}{\sqrt{n^2+1}}.$$

To figure out whether this series converges, we use a comparison test. To figure out what to compare to, we look at the dominant terms in the expression n^2+1. As n becomes large, the dominant term is n^2, so

$$\frac{1}{\sqrt{n^2+1}} \approx \frac{1}{\sqrt{n^2}} = \frac{1}{n}.$$

We know that $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, so we expect $\sum_{n=0}^{\infty}\frac{1}{\sqrt{n^2+1}}$ to diverge as well. To know for sure, we should use a comparison test.

We cannot use the regular comparison test because

$$\frac{1}{\sqrt{n^2+1}} \leq \frac{1}{\sqrt{n^2}} = \frac{1}{n}.$$

Knowing that we are smaller than a divergent series doesn't tell us anything. But we can use the limit comparison test.

$$\lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{1}{\sqrt{n^2+1}}} = \lim_{n\to\infty}\sqrt{\frac{n^2+1}{n^2}} = \sqrt{1} = 1.$$

Since $0 < 1 < \infty$, we have a valid limit comparison, so from the fact that $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges we can conclude that $\sum_{n=0}^{\infty}\frac{1}{\sqrt{n^2+1}}$ diverges as well.

When $x = 2$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{2^n \sqrt{n^2+1}} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{\sqrt{n^2+1}}.$$

This is an alternating series. The alternating series test tells us to compute $\lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}}$. As n becomes large, the denominator becomes large, so the limit is zero. Hence, this series converges by the alternating series test.

Thus, the series $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^n \sqrt{n^2+1}}$ converges when $-2 < x \leq 2$.

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3n}(x+3)^n$

Solution: We apply the ratio test. We see that

$$a_n = \frac{2^n}{3n}(x+3)^n, \qquad a_{n+1} = \frac{2^{n+1}}{3(n+1)}(x+3)^{n+1}.$$

Next, we compute that

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{\frac{2^{n+1}}{3(n+1)}(x+3)^{n+1}}{\frac{2^n}{3n}(x+3)^n}\right| \\ &= \left|\frac{2^{n+1}}{2^n} \cdot \frac{3n}{3(n+1)} \cdot \frac{(x+3)^{n+1}}{(x+3)^n}\right| \\ &= 2 \cdot \frac{n}{n+1} \cdot |x+3|.\end{aligned}$$

Thus,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 2 \cdot 1 \cdot |x+3| = 2|x+3|.$$

The ratio test tells us that this series converges when $2|x+3| < 1$ and diverges when $2|x+3| > 1$. When $2|x+3| = 1$, that is, when $x+3 = \pm\frac{1}{2}$, we need to investigate further.

When $x+3 = -\frac{1}{2}$, we have the series

$$\sum_{n=1}^{\infty} \frac{2^n}{3n}\left(-\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} (-1)^n \frac{1}{3n}.$$

This is an alternating series. Since $\lim_{n\to\infty} \frac{1}{3n}$, the series converges by the alternating series test.

When $x+3 = \frac{1}{2}$, we have the series

$$\sum_{n=1}^{\infty} \frac{2^n}{3n}\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{1}{3n} = \frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n}.$$

The harmonic series diverges, so this series diverges.

We conclude that the series $\sum_{n=1}^{\infty} \frac{2^n}{3n}(x+3)^n$ converges when $-\frac{1}{2} \leq x+3 < \frac{1}{2}$, that is, when $-\frac{7}{2} \leq x < -\frac{5}{2}$.

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{(-5)^n}{n!}(x+10)^n$

Solution: Applying the ratio test, we compute that

$$a_n = \frac{(-5)^n}{n!}(x+10)^n, \qquad a_{n+1} = \frac{(-5)^{n+1}}{(n+1)!}(x+10)^{n+1}.$$

Then,

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{\frac{(-5)^{n+1}}{(n+1)!}(x+10)^{n+1}}{\frac{(-5)^n}{n!}(x+10)^n}\right| \\ &= \frac{5^{n+1}}{5^n}\cdot\frac{n!}{(n+1)!}\cdot\left|\frac{(x+10)^{n+1}}{(x+10)^n}\right| \\ &= 5\cdot\frac{1}{n+1}\cdot|x+10|\,.\end{aligned}$$

Thus,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 5\cdot 0\cdot|x+10| = 0.$$

Since $0<1$, the ratio test tells us that the series always converges, that is, for $-\infty<x<\infty$.

(4) $\displaystyle\sum_{n=1}^{\infty} e^n(x-2)^n$

Solution: This series is a geometric series with common ration $e(x-2)$. Thus, it converges when $|e(x-2)|<1$, which happens when $|x-2|<\frac{1}{e}$, which happens when $2-\frac{1}{e}<x<2+\frac{1}{e}$.

We could also use the root test or the ratio test. The root test asks us to compute

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{e^n\,|x-2|^n} = \lim_{n\to\infty} e\,|x-2| = e\,|x-2|\,.$$

Thus, the series converges when $e\,|x-2|<1$ and diverges when $e\,|x-2|>1$. That is, the series converges when $|x-2|<\frac{1}{e}$, and when $x-2=\pm\frac{1}{e}$, we need to investigate further.

When $x-2=-\frac{1}{e}$, the series becomes

$$\sum_{n=1}^{\infty} e^n\left(-\frac{1}{e}\right)^n = \sum_{n=1}^{\infty}(-1)^n,$$

which diverges.

When $x-2=\frac{1}{e}$, the series becomes

$$\sum_{n=1}^{\infty} e^n\left(\frac{1}{e}\right)^n = \sum_{n=1}^{\infty} 1.$$

This series also diverges.

Thus, as before, we find that the series $\sum_{n=1}^{\infty} e^n(x-2)^n$ converges when $-\frac{1}{e}<x-2<\frac{1}{e}$, that is, when $2-\frac{1}{e}<x<2+\frac{1}{e}$.

11.9: Representing Functions as Power Series.

(1) Represent the following functions as power series (centered at $x=0$), and write down the radius of convergence.

(a) $f(x)=\dfrac{x-1}{x+2}$

Solution: We know that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,$$

which converges when $|x|<1$. Then,

$$\begin{aligned}\frac{1}{x+2} &= \frac{1}{2}\cdot\frac{1}{1+\frac{x}{2}} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{1}{2^{n+1}}x^n.\end{aligned}$$

Since we substituted $-\frac{x}{2}$ for x, we know that this series converges when $\left|-\frac{x}{2}\right| < 1$, that is, when $|x| < 2$. Then,

$$\frac{x}{x+2} = x\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^{n+1}}x^n = \sum_{n=0}^{\infty}(-1)^n\frac{1}{2^{n+1}}x^{n+1}.$$

Multiplying every term in the series by the same number (in this case x) does not change whether a series converges, so the interval of convergence is still $|x| < 2$. Shifting the terms over, we can write the same series as

$$\frac{x}{x+2} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{2^n}x^n.$$

Next, we know that

$$\frac{-1}{x+2} = -\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^{n+1}}x^n = \sum_{n=0}^{\infty}(-1)^{n-1}\frac{1}{2^{n+1}}x^n.$$

Again, multiplying every term in the series by the same number (in this case -1) does not change whether the series converges, so the interval of convergence is also $|x| < 2$. We can separate out the first term and write the same series as

$$\frac{-1}{x+2} = -\frac{1}{2} + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{2^{n+1}}x^n.$$

Adding the two together, we find that

$$\begin{aligned}\frac{x-1}{x+2} &= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{2^n}x^n + \left(-\frac{1}{2} + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{2^{n+1}}x^n\right)\\ &= -\frac{1}{2} + \sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{2^n} + \frac{1}{2^{n+1}}\right)x^n\\ &= -\frac{1}{2} + \sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{3}{2^{n+1}}\right)x^n.\end{aligned}$$

Adding two series that converge when $|x| < 2$ gives another series that converges when $|x| < 2$, so the radius of convergence is 2. If you find the summation notation difficult to follow, remember that you can always gain understanding by writing out the first few terms of each sum.
Alternatively, using long division we can find that

$$\begin{aligned}\frac{x-1}{x+2} &= 1 - \frac{3}{x+2}\\ &= 1 - 3\left(\frac{1}{x+2}\right)\\ &= 1 - 3\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^{n+1}}x^n\\ &= 1 + \sum_{n=0}^{\infty}(-1)^{n-1}\frac{3}{2^{n+1}}x^n.\end{aligned}$$

By separating out the first term of the series and combining it with the 1, we obtain the same answer as before.

(b) $f(x) = \dfrac{2x-4}{x^2-4x+3}$ (Hint: First use partial fractions.)
Solution: Using the method of partial fractions, we find that

$$\frac{2x-4}{x^2-4x+3} = \frac{1}{x-1} + \frac{1}{x-3}.$$

Working on the first term, we find that

$$\frac{1}{x-1} = -\frac{1}{1-x} = -\sum_{n=0}^{\infty} x^n.$$

We know that this series converges when $|x| < 1$.
For the second term, we see that

$$\begin{aligned}\frac{1}{x-3} &= -\frac{1}{3} \cdot \frac{1}{1-\frac{x}{3}} \\ &= -\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n \\ &= -\sum_{n=0}^{\infty}\frac{1}{3^{n+1}}x^n.\end{aligned}$$

This series converges when $\left|\frac{x}{3}\right| < 1$, that is, when $|x| < 3$.
Adding the two, we find that

$$\frac{2x-4}{x^2-4x+3} = -\sum_{n=0}^{\infty}\left(1+\frac{1}{3^{n+1}}\right)x^n.$$

We added together a series that converges when $|x| < 1$ with a series that converges when $|x| < 3$. Both series need to convere in order for the sum to converge, so our series converges when $|x| < 1$, and so the radius of convergence is 1.
Why is the sum negative? We can see that $f(0) = \frac{-4}{3}$, which is negative. We haven't computed the interval of convergence, but we know that it's going to be some interval centered at $x = 0$. Since $f(x)$ is negative when $x = 0$, it is also going to be negative for values of x that are near zero.

(c) $f(x) = \dfrac{x^3}{(1-3x)^2}$

Solution: In class, we took the derivative of $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ to find that

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1}.$$

Taking the derivative does not change the radius of convergence, so the radius of convergence of this series is still 1. Substituting $3x$ for x, we find that

$$\frac{1}{(1-3x)^2} = \sum_{n=0}^{\infty} n(3x)^{n-1} = \sum_{n=0}^{\infty} n3^{n-1}x^{n-1}.$$

Ignoring the issue of endpoints since we're only interested in the radius of convergence, since the series for $\frac{1}{(1-x)^2}$ converges when $|x| < 1$, the series for $\frac{1}{(1-3x)^2}$ converges when $|3x| < 1$, that is, when $|x| < \frac{1}{3}$.
Next, we multiply by x^3. We find that

$$\frac{x^3}{(1-3x)^2} = \sum_{n=0}^{\infty} n3^{n-1}x^{n+2}.$$

Multiplying every term of the series by the same number (in this case x^3) does not change whether or not the series converges, so (ignoring the issue of endpoints), this series converges when $|x| < \frac{1}{3}$, so the radius of convergence is $\frac{1}{3}$.

There are several ways of rewriting this series. Write out the first few terms to see why.

$$\begin{aligned}\frac{x^3}{(1-3x)^2} &= \sum_{n=0}^{\infty} n3^{n-1}x^{n+2} \\ &= \sum_{n=1}^{\infty} n3^{n-1}x^{n+2} \\ &= \sum_{n=3}^{\infty} (n-2)3^{n-3}x^n.\end{aligned}$$

(d) $f(x) = \ln\left(1+x^4\right)$

Solution: By integrating $\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$, we saw in class that

$$\begin{aligned}\ln(1+x) &= \sum_{n=0}^{\infty}(-1)^n \frac{1}{n+1}x^{n+1} \\ &= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}x^n.\end{aligned}$$

As we saw in class, the radius of convergence of this series is 1 because integration does not change the radius of convergence.

Next, we substitute x^4 for x to find that

$$\begin{aligned}\ln(1+x^4) &= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}\left(x^4\right)^n \\ &= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}x^{4n}.\end{aligned}$$

Ignoring the endpoint issue since we're only looking for the radius of convergence, we see that since the series fo $\ln(1+x)$ converges when $|x| < 1$, the series for $\ln(1+x^4)$ converges when $\left|x^4\right| < 1$, which happens exactly when $|x| < 1$. Thus, the radius of convergence of this series is 1.

(2) Evaluate the following indefinite integrals as power series. Write down the radius of convergence.

(a) $\displaystyle\int \frac{t}{1-t^8}\,dt$

Solution: From class, we know that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, with a radius of convergence of 1. Plugging in t^8 for x, we find that

$$\frac{1}{1-t^8} = \sum_{n=0}^{\infty}\left(t^8\right)^n = \sum_{n=0}^{\infty} t^{8n}.$$

Since the series for $\frac{1}{1-x}$ converges when $|x| < 1$, the series for $\frac{1}{1-t^8}$ converges when $\left|t^8\right| < 1$, which happens when $|t| < 1$. Thus, the radius of convergence of this series is still 1.

Multiplying by t, we find that

$$\frac{t}{1-t^8} = t\sum_{n=0}^{\infty} t^{8n} = \sum_{n=0}^{\infty} t^{8n+1}.$$

Multiplying every term of the series by the same number, in this case t, does not change whether the series converges, so the radius of convergence of this series is still 1. Integrating, we find that

$$\int \frac{t}{1-t^8} = C + \sum_{n=0}^{\infty}\frac{1}{8n+2}t^{8n+2}.$$

Since integration does not change the radius of convergence, the radius of convergence of this series is still 1.

(b) $\displaystyle\int x^2\ln(1+x)\,dx$

Solution: In class, we used integration of a power series to find out that

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n$$

and that this series has a radius of convergence of 1. Next, we write that

$$x^2 \ln(1+x) = x^2 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^{n+2}.$$

Multiplying every term of the series by the same number, in this case x^2, does not change whether or not it converges, so the radius of convergence of this series is still 1. Integrating, we find that

$$\int x^2 \ln(1+x) = C + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} \cdot \frac{1}{n+3} x^{n+3}$$

Integration does not change the radius of convergence, so the radius of convergence is still 1. There are a few other ways of expressing this answer, such as

$$\begin{aligned} \int x^2 \ln(1+x) &= C + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n(n+3)} x^{n+3} \\ &= \sum_{n=4}^{\infty} (-1)^{n} \frac{1}{(n-3)n} x^{n}. \end{aligned}$$

11.10: Taylor and Maclaurin series.

(1) Find the first four nonzero terms of the Taylor series for $f(x)$ centered at the given value.

(a) $f(x) = \ln x$ centered at 1

Solution: To find the terms of the Taylor series, we need to compute the derivatives of f. We need four nonzero values. We compute

$$\begin{array}{lllll} f(x) = \ln(x), & f'(x) = \frac{1}{x}, & f''(x) = -\frac{1}{x^2}, & f^{(3)}(x) = \frac{2}{x^3}, & f^{(4)}(x) = -\frac{6}{x^4}, \\ f(1) = 0, & f'(1) = 1, & f''(1) = -1, & f^{(3)}(1) = 2, & f^{(4)}(1) = -6. \end{array}$$

We know that the relationship between the coefficients c_n of the Taylor series centered at a and the derivatives of f at a is $c_n = \frac{f^{(n)}(a)}{n!}$, so we can compute that

$$\begin{aligned} \ln x &\approx \frac{0}{0!} + \frac{1}{1!}(x-1) - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 - \frac{6}{4!}(x-1)^4 \\ &= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4. \end{aligned}$$

Note that substituting $1+x$ for x we find that

$$\ln(1+x) \approx x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4,$$

which we already knew from before.

(b) $f(x) = x^{\frac{1}{3}}$ centered at 8

Solution: Again, we compute the derivatives of f. We need four nonzero values. To evaluate effectively at $x = 8$, remember to first take the cube root and then exponentiate rather than the other way around.

$$\begin{array}{llll} f(x) = x^{1/3}, & f'(x) = \frac{1}{3}x^{-2/3}, & f''(x) = -\frac{2}{9}x^{-5/3}, & f'''(x) = \frac{10}{27}x^{-8/3}, \\ f(8) = 2, & f'(8) = \frac{1}{3} \cdot \frac{1}{4}, & f''(8) = -\frac{2}{9} \cdot \frac{1}{32}, & f'''(x) = \frac{10}{27}\frac{1}{256} \end{array}$$

Instead of simplifying, we write down these values in a helpful form.

$$\begin{array}{llll} f(8) = 2, & f'(8) = \frac{1}{12}, & f''(8) = -\frac{1}{9 \cdot 16}, & f'''(x) = \frac{5}{3^3 \cdot 2^7} \end{array}$$

Using the fact that $c_n = \frac{f^{(n)}(a)}{n!}$, we can write down the Taylor series as

$$x^{1/3} \approx 2 + \frac{1}{12}(x-8) - \frac{1}{9 \cdot 2^5}(x-8)^3 + \frac{5}{3^4 \cdot 2^8}(x-8)^3.$$

(c) $f(x) = \sin x$ centered at $\frac{\pi}{6}$.
Solution: We compute the derivatives of f. We need four nonzero terms.

$$\begin{aligned}
&f(x) = \sin x, && f'(x) = \cos x, && f''(x) = -\sin x, && f'''(x) = -\cos x,\\
&f(\tfrac{\pi}{6}) = \tfrac{1}{2}, && f'(\tfrac{\pi}{6}) = \tfrac{\sqrt{3}}{2}, && f''(\tfrac{\pi}{6}) = -\tfrac{1}{2}, && f'''(x) = -\tfrac{\sqrt{3}}{2}.
\end{aligned}$$

Next, using the relationship $c_n = \frac{f^{(n)}(a)}{n!}$ for the coefficients of the Taylor series, we find that

$$\sin x \approx \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}(x - \tfrac{\pi}{6}) - \tfrac{1}{4}(x - \tfrac{\pi}{6})^2 - \tfrac{\sqrt{3}}{12}(x - \tfrac{\pi}{6})^3.$$

(d) $f(x) = e^x \cos x$ centered at 0. You can solve this problem either by computing the derivatives of f as usual or by multiplying the Maclaurin series for e^x and $\cos x$ together.
Solution: Computing derivatives, we find that

$$\begin{aligned}
f(x) &= e^x \cos x, & f(0) &= 1,\\
f'(x) &= e^x \cos x - e^x \sin x = e^x(\cos x - \sin x), & f'(0) &= 1,\\
f''(x) &= e^x(\cos x - \sin x) + e^x(-\sin x - \cos x) = -2e^x \sin x, & f''(0) &= 0,\\
f'''(x) &= -2e^x \sin x - 2e^x \cos x = -2e^x(\cos x + \sin x), & f'''(0) &= -2,\\
f^{(4)}(x) &= -2e^x(\cos x + \sin x) - 2e^x(-\sin x + \cos x) = -4e^x \cos x, & f^{(4)}(0) &= -4.
\end{aligned}$$

Using $c_n = \frac{f^{(n)}(a)}{n!}$, we see that the Taylor polynomial is

$$\begin{aligned}
e^x \cos x &\approx \tfrac{1}{1} + \tfrac{1}{1}x + \tfrac{0}{2}x^2 - \tfrac{2}{6}x^3 - \tfrac{4}{24}x^4\\
&= 1 + x - \tfrac{1}{3}x^3 - \tfrac{1}{6}x^4.
\end{aligned}$$

Alternatively, we multiply together the following two series

$$\begin{aligned}
e^x &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots\\
\cos x &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots
\end{aligned}$$

We see that the constant term of $e^x \cos x$ is $1 \cdot 1 = 1$ and the x term is $x \cdot 1 = x$. The x^2 term is

$$\frac{1}{2}x^2 \cdot 1 + 1 \cdot \left(-\frac{1}{2}x^2\right) = 0.$$

The x^3 term is

$$\frac{1}{6}x^3 \cdot 1 + x \cdot \left(-\frac{1}{2}x^2\right) = -\frac{1}{3}x^3.$$

The x^4 term is

$$\frac{1}{24}x^4 \cdot 1 + \frac{1}{2}x^2 \cdot \left(-\frac{1}{2}x^2\right) + 1 \cdot \frac{1}{24}x^4 = -\frac{1}{6}x^4.$$

Putting these terms together, we get the same answer as before.

(2) Find the Maclaurin series for $f(x)$. Use 11.9 techniques, 11.10 techniques, or a combination of the two.

(a) $f(x) = \frac{2x^2}{(1-4x)^2}$

Solution: In class, we differentiated $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ to find that

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1}.$$

Plugging in $4x$ for x, we find that

$$\frac{1}{(1-4x)^2} = \sum_{n=0}^{\infty} n(4x)^{n-1} = \sum_{n=0}^{\infty} n4^{n-1}x^{n-1}.$$

Multiplying by $2x^2$, we find that

$$\frac{2x^2}{(1-4x)^2} = 2x^2 \sum_{n=0}^{\infty} n4^{n-1}x^{n-1} = \sum_{n=0}^{\infty} 2n \cdot 4^{n-1}x^{n+1}.$$

There are other ways of writing the answer, such as

$$\begin{aligned}\frac{2x^2}{(1-4x)^2} &= \sum_{n=1}^{\infty} 2n \cdot 4^{n-1}x^{n+1} \\ &= \sum_{n=2}^{\infty} 2(n-1) \cdot 4^{n-2}x^n \\ &= \sum_{n=2}^{\infty} (n-1)2^{2n-3}x^n.\end{aligned}$$

(b) $f(x) = \frac{1}{(1-x)^3}$
Solution: We could use a Taylor series for this one, but it's easier to just go directly, In class, we differentiated $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ to find that

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1}.$$

We were able to use this expression to find series for similar expressions that had things like $(1-4x)^2$ or $(4-x)^2$ in the denominator. But now we need a cube in the numerator, so we can differentiate again to find that

$$\frac{2}{(1-x)^3} = \sum_{n=0}^{\infty} n(n-1)x^{n-2}.$$

Thus,

$$\frac{1}{(1-x)^3} = \frac{1}{2}\sum_{n=0}^{\infty} n(n-1)x^{n-2}.$$

The first two terms of this series are zero, so we can remove those terms and shift the terms over to rewrite it as

$$\begin{aligned}\frac{1}{(1-x)^3} &= \frac{1}{2}\sum_{n=2}^{\infty} n(n-1)x^{n-2} \\ &= \frac{1}{2}\sum_{n=0}^{\infty} (n+2)(n+1)x^n.\end{aligned}$$

(c) $f(x) = 2^x$

Solution: If we want to write down the Taylor series, we need to be able to differentiate 2^x. To do so, we note that $\ln(2^x) = x \ln 2$, and so

$$2^x = e^{x \ln 2}.$$

At this point we could proceed with writing down the Taylor series, but at this point it's easiest to take the Taylor series of e^x from class and substitute in $x \ln 2$ for x. We have that

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n,$$

so

$$\begin{aligned} 2^x = e^{x \ln 2} &= \sum_{n=0}^{\infty} \frac{1}{n!} (x \ln 2)^n \\ &= \sum_{n=0}^{\infty} \frac{(\ln 2)^n}{n!} x^n. \end{aligned}$$

If we were to proceed with the Taylor series, we'd find that $f(x) = e^{x \ln 2}$, $f'(x) = \ln 2 e^{x \ln 2}$, $f''(x) = (\ln 2)^2 e^{x \ln 2}$, $f^{(3)}(x) = (\ln 2)^3 e^{x \ln 2}$, and so forth. In general, we'd find that

$$\begin{aligned} f^{(n)}(x) &= (\ln 2)^n e^{x \ln 2} \\ f^{(n)}(0) &= (\ln 2)^n. \end{aligned}$$

Constructing our Taylor series with the formula $\sum_{n=0}^{\infty} \frac{f^{(n)}}{n!} x^n$, we'd obtain the same answer as before.

(d) $f(x) = x \cos(2x)$

Solution: Again, we could proceed directly by computing a Taylor series, but it's easiest to instead compute the Taylor series of $\cos(2x)$ and then multiply the result by x. For $\cos(2x)$, again we could compute the Taylor series or we could use the Taylor series for $\cos x$ and plug in $2x$ for x. The formula for the Taylor series for $\cos x$ is in the book, but you can also find it by computing the derivatives: $\cos x$, $-\sin x$, $-\cos x$, $\sin x$, $\cos x$, $-\sin x$, and so forth, which when evaluated at $x = 0$, give $1, 0, -1, 0, 1, 0, \ldots$. In any case, however you go about it, you find that

$$\begin{aligned} \cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n} \\ \cos 2x &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} (2x)^{2n} = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}}{(2n)!} x^{2n} \\ x \cos 2x &= x \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}}{(2n)!} x^{2n} = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}}{(2n)!} x^{2n+1}. \end{aligned}$$

Again, to understand the solution it's always helpful to write out the first few terms of each summation expression.

(3) Find the Taylor series for $f(x)$ with the given center. Again, use any techniques.

(a) $f(x) = \ln x$ centered at 2

Solution: We compute the first few derivatives of $\ln x$. We find that

$$\begin{aligned} f(x) &= \ln x, \\ f'(x) &= \frac{1}{x}, \\ f''(x) &= -\frac{1}{x^2}, \\ f^{(3)}(x) &= 2\frac{1}{x^3}, \\ f^{(4)}(x) &= -6\frac{1}{x^4}, \\ f^{(5)}(x) &= 24\frac{1}{x^5}. \end{aligned}$$

It looks like the expressions are alternating, and each step we multiply by the next integer giving us a factorial expression. Understanding the pattern, we find that for $n \geq 1$,

$$f^{(n)}(x) = (-1)^{n-1}(n-1)!\frac{1}{x^n}.$$

When $n = 0$ we of course have $f^{(0)}(x) = f(x) = \ln x$.
We are looking for the Taylor series centered at $x = 2$, so we plug in $x = 2$ to find that, for $n \geq 1$.

$$f^{(n)}(2) = (-1)^{n-1}(n-1)!\frac{1}{2^n}.$$

Again, for $n = 0$ this formula does not apply and we instead have $f^{(0)}(2) = f(2) = \ln 2$. Thus, if we want to write the Taylor series using summation notation, we'll need to address the first term separately, since it doesn't fit the pattern. The Taylor series is

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n &= f(2) + \sum_{n=1}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n \\ &= \ln 2 + \sum_{n=1}^{\infty} (-1)^{n-1}\frac{(n-1)!}{2^n n!}(x-2)^n \\ &= \ln 2 + \sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{2^n n}(x-2)^n. \end{aligned}$$

(b) $f(x) = x^3 + 3x^2 + x$ centered at 2. What is the interval of convergence of the Taylor series?
Solution: We compute the derivatives of $f(x)$ and evaluate them at $x = 2$. We have

$$\begin{array}{llllll} f(x) = x^3 + 3x^2 + x, & f'(x) = 3x^2 + 6x + 1, & f''(x) = 6x + 6, & f^{(3)}(x) = 6, & f^{(4)}(x) = 0, & f^{(5)}(x) = 0. \\ f(2) = 22, & f'(2) = 25, & f''(2) = 18, & f^{(3)}(2) = 6, & f^{(4)}(2) = 0, & f^{(5)}(2) = 0. \end{array}$$

We see that all remaining derivatives of f will be zero. Thus, writing the Taylor series, we find that

$$\begin{aligned} x^3 + 3x^2 + 2 &= 22 + 25(x-2) + \frac{18}{2}(x-2)^2 + \frac{6}{3!}(x-2)^3 + \frac{0}{4!}(x-2)^4 + \frac{0}{5!}(x-2)^5 + \cdots \\ &= 22 + 25(x-2) + 9(x-2)^2 + (x-2)^3. \end{aligned}$$

This infinte series is actually finite, and finite sums always make sense, so the interval of convergence is $-\infty < x < \infty$. More precisely, it is an infinite series where all of the terms except for the first four are zero. That means that, after the first four, all of the partial sums are the same. A sequence that is eventually constant converges.

(c) $f(x) = \sin(2x)$ centered at π

Solution: There are a couple strategies for this one, but the most straightforward is to compute

$$\begin{array}{lllll} f(x) = \sin(2x), & = f'(x) = 2\cos(2x), & = f''(x) = -4\sin(2x), & f^{(3)}(x) = -8\cos(2x), & f^{(4)}(x) = 16\sin(2x). \\ f(\pi) = 0, & f'(\pi) = 2, & f''(\pi) = 0, & f^{(3)}(\pi) = -8, & f^{(4)}(\pi) = 0. \\ c_0 = 0, & c_1 = 2, & c_3 = 0, & c_4 = -\dfrac{8}{3!}, & c_4 = 0. \end{array}$$

Thus,

$$f(x) \approx 2(x-\pi) - \frac{2^3}{3!}(x-\pi)^3.$$

At this point we see that things are going to keep repeating every four derivatives, getting bigger by a factor of 16 each time. We could also write out a few more terms to get a better sense of the pattern. We can also compare and contrast the numbers we're getting to the numbers we got in the Taylor series of $\sin x$. We see that there are some powers of 2, but everything else is the same. However we go about it, we can find the pattern and write the full Taylor series

$$\sin 2x = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1}}{(2n+1)!}(x-\pi)^{2n+1}.$$

The faster way to go about it is to start from the series for $\sin x$ at $x = 0$ and working our way up to answering the question. The key thing to do is to understand what $\sin 2x$ looks like near $x = \pi$. It looks just the same as $\sin 2x$ near $x = 0$. Why is that? We know that sine has a period of 2π, so $\sin 2x = \sin(2x - 2\pi) = \sin(2(x-\pi))$.
Using Taylor series or by looking it up in the book, we see that

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}.$$

Then,

$$\begin{aligned} \sin 2x &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!}(2x)^{2n+1} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1}}{(2n+1)!} x^{2n+1}. \end{aligned}$$

This is a power series centered at $x = 0$. We need a power series centered at $x = \pi$. We can do that by substituting $x - \pi$ for x. We get that

$$\sin 2(x-\pi) = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1}}{(2n+1)!}(x-\pi)^{2n+1}.$$

But as discussed above, $\sin 2(x-\pi) = \sin 2x$, so that's our answer.

$$\sin 2x = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1}}{(2n+1)!}(x-\pi)^{2n+1}.$$

(d) $f(x) = x^2 \ln(1 + x^3)$ centered at 0
Solution: It's best to do this one starting from the series of $\ln(1+x)$, which we found in class by integrating $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$. We found that

$$\begin{aligned} \ln(1+x) &= \sum_{n=0} (-1)^n \frac{1}{n+1} x^{n+1} \\ &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n. \end{aligned}$$

Either form is fine, but we'll work with the second one for simplicity. Plugging in x^3 for x, we find that

$$\begin{aligned}\ln(1+x^3) &= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}\left(x^3\right)^n \\ &= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}x^{3n}.\end{aligned}$$

Multiplying by x^2, we find that

$$\begin{aligned}x^2\ln(1+x^3) &= x^2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}x^{3n} \\ &= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}x^{3n+2}.\end{aligned}$$

(4) Evaluate the indefinite integral as an infinite series. Again, use any techniques.

(a) $\displaystyle\int \sin\left(x^3\right)\ dx$

Solution: We start with the power series for $\sin x$.

$$\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}x^{2n+1}.$$

Next, we plug in x^3 for x. We find that

$$\begin{aligned}\sin x^3 &= \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}\left(x^3\right)^{2n+1} \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}x^{6n+3}.\end{aligned}$$

Finally, we integrate

$$\int \sin\left(x^3\right)\ dx = C + \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}\cdot\frac{1}{6n+4}x^{6n+4}.$$

(b) $\displaystyle\int \frac{\cos x - 1}{x}\ dx$

Solution: For this one, it's definitely a good idea to write down the terms of the series first and only switch to sum notation after getting a good idea of what's going on. We have that

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 \pm \cdots.$$

Thus,

$$\cos x - 1 = -\frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 \pm \cdots.$$

We see that things work out nicely on the right-hand side when we divide by x.

$$\frac{\cos x - 1}{x} = -\frac{1}{2}x + \frac{1}{4!}x^3 - \frac{1}{6!}x^5 \pm \cdots.$$

Finally, we integrate.

$$\int \frac{\cos x - 1}{x}\ dx = C - \frac{1}{2}\cdot\frac{1}{2}x^2 + \frac{1}{4!}\cdot\frac{1}{4}x^4 - \frac{1}{6!}\cdot\frac{1}{6}x^6 \pm \cdots.$$

It is not hard to figure out the pattern and write the answer using sum notation from here, but let's go back to the beginning and go through it step by step using sum notation.

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}.$$

$$\cos x - 1 = \sum_{n=1}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}.$$

$$\frac{\cos x - 1}{x} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n-1}.$$

$$\int \frac{\cos x - 1}{x}\, dx = C + \sum_{n=1}^{\infty} (-1)^n \frac{1}{(2n)!} \cdot \frac{1}{2n} x^{2n}.$$

It's not a bad idea to take this approach to any problem when studying: First solve the problem writing out the first few terms, and then go through the same steps using sum notation.