Math 132: Discussion Session: Week 9

Directions: In groups of 3-4 students, work the problems on the following page. Below, list the members of your group and your answers to the specified questions. Turn **this paper** in at the end of class. You do not need to turn in the question page or your work.

Additional Instructions: It is okay if you do not completely finish all of the problems. Also, each group member should work through each problem, as similar problems may appear on the exam.

Scoring:

Correct answers	Grade
0–3	0%
4–6	80%
7–11	100%

Group Members:

8.2: Surface Area.

(1) The area is:

(2) The area is:

(3) The area is:

(4) The area is:

11.1: Sequences.

(1) $\lim_{n\to\infty} a_n =$

(2) $\lim_{n\to\infty} a_n =$

(3) $\lim_{n\to\infty} a_n =$

(4) $\lim_{n\to\infty} a_n =$

(5) $\lim_{n\to\infty} a_n =$

(6) $\lim_{n\to\infty} a_n =$

(7) $\lim_{n\to\infty} a_n =$

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8.2: Surface Area. Find the exact area of the surface obtained by rotating the given curve about the given axis.

(1) $y = x^3$ from $x = 0$ to $x = 1$, rotated about the x-axis

Solution: Since we are rotating about the x-axis, the radius of the circles is y, so the surface area is

$$\begin{aligned}\int_{x=0}^{1} 2\pi y\, ds &= 2\pi \int_0^1 y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \\ &= 2\pi \int_0^1 x^3\sqrt{1+(3x^2)^2}\, dx \\ &= 2\pi \int_0^1 x^3\sqrt{1+9x^4}\, dx.\end{aligned}$$

We can compute the integral with the substitution $u = 1 + 9x^4$, $du = 36x^3\, dx$. The new bounds are $u(0) = 1$ and $u(1) = 10$. We find that

$$\begin{aligned}2\pi \int_0^1 x^3\sqrt{1+9x^4}\, dx &= \frac{\pi}{18}\int_{x=0}^{1} \sqrt{1+9x^4}\cdot 36x^3\, dx \\ &= \frac{\pi}{18}\int_{u=1}^{10} \sqrt{u}\, du \\ &= \frac{\pi}{18}\left(\tfrac{2}{3}u^{3/2}\right)\Big|_{u=1}^{10} \\ &= \boxed{\frac{\pi}{27}\left(10\sqrt{10}-1\right)}\end{aligned}$$

(2) $y = \sqrt{1+e^x}$ from $x = 0$ to $x = 2$, rotated about the x-axis.

Solution: Since we are rotating about the x-axis, the radius of the circles is y, so the surface area is

$$\begin{aligned}\int_{x=0}^{2} 2\pi y\, ds &= 2\pi \int_0^2 y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \\ &= 2\pi \int_0^2 \sqrt{1+e^x}\sqrt{1+\left(\frac{1}{2\sqrt{1+e^x}}\cdot e^x\right)^2}\, dx \\ &= 2\pi \int_0^2 \sqrt{1+e^x}\sqrt{1+\frac{e^{2x}}{4(1+e^x)}}\, dx \\ &= 2\pi \int_0^2 \sqrt{(1+e^x)\left(1+\frac{e^{2x}}{4(1+e^x)}\right)}\, dx \\ &= 2\pi \int_0^2 \sqrt{(1+e^x)+\frac{e^{2x}}{4}}\, dx.\end{aligned}$$

To simplify further, we need to factor the expression under the square root.

$$\begin{aligned}
2\pi \int_0^2 \sqrt{(1+e^x) + \frac{e^{2x}}{4}}\, dx &= \pi \int_0^2 \sqrt{4 + 4e^x + e^{2x}}\, dx \\
&= \pi \int_0^2 \sqrt{(2+e^x)^2}\, dx \\
&= \pi \int_0^2 (2+e^x)\, dx \\
&= \pi \left(2x + e^x\right)\Big|_{x=0}^{2} \\
&= \pi \left(2(2-0) + (e^2 - e^0)\right) \\
&= \boxed{\pi \left(3 + e^2\right).}
\end{aligned}$$

(3) $y = \frac{1}{3}x^{\frac{3}{2}}$ $0 \le x \le 16$, rotated about the y-axis.

Solution: Since we are rotating about the y-axis, the radius of the circles is x, so the surface area is

$$\begin{aligned}
\int_{x=0}^{16} 2\pi x\, ds &= 2\pi \int_0^{16} x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
&= 2\pi \int_0^{16} x\sqrt{1 + \left(\tfrac{1}{2}x^{1/2}\right)^2}\, dx \\
&= 2\pi \int_0^{16} x\sqrt{1 + \tfrac{1}{4}x}\, dx.
\end{aligned}$$

To evaluate this integral, we do the substitution $u = 1 + \frac{1}{4}x$, $du = \frac{1}{4}\, dx$. To perform the substitution, we will need to solve for x in terms of u, and we find that $x = 4(u-1)$. The new bounds are $u(0) = 1$ and $u(16) = 5$. We compute that

$$\begin{aligned}
2\pi \int_0^{16} x\sqrt{1 + \tfrac{1}{4}x}\, dx &= 8\pi \int_0^{16} x\sqrt{1 + \tfrac{1}{4}x} \cdot \tfrac{1}{4}\, dx \\
&= 8\pi \int_{u=1}^{5} 4(u-1)\sqrt{u}\, du \\
&= 32\pi \int_{u=1}^{5} \left(u^{3/2} - u^{1/2}\right)\, du \\
&= 32\pi \left(\tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2}\right)\Big|_{u=1}^{5} \\
&= 32\pi \left(\left(\tfrac{2}{5}(25\sqrt{5}) - \tfrac{2}{3}(5\sqrt{5})\right) - \left(\tfrac{2}{5} - \tfrac{2}{3}\right)\right) \\
&= 32\pi \left(\tfrac{20}{3}\sqrt{5} + \tfrac{4}{15}\right) \\
&= \boxed{\left(\frac{640\sqrt{5}}{3} + \frac{128}{15}\right)\pi}
\end{aligned}$$

(4) $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x$, $1 \le x \le 3$, rotated about the y-axis.

Solution: Since we are rotating about the y-axis, the radius of the circles is x, so the surface area is

$$\begin{aligned}
\int_{x=1}^{3} 2\pi x\, ds &= 2\pi \int_1^3 x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \\
&= 2\pi \int_1^3 x\sqrt{1+\left(\tfrac{1}{2}x - \tfrac{1}{2x}\right)^2}\, dx \\
&= 2\pi \int_1^3 x\sqrt{1+\tfrac{1}{4}\left(x^2 - 2 + \tfrac{1}{x^2}\right)}\, dx \\
&= 2\pi \int_1^3 x\sqrt{\tfrac{1}{4}\left(x^2 + 2 + \tfrac{1}{x^2}\right)}\, dx \\
&= 2\pi \int_1^3 x\sqrt{\tfrac{1}{4}\left(x + \tfrac{1}{x}\right)^2}\, dx \\
&= 2\pi \int_1^3 x\left(\tfrac{1}{2}\left(x + \tfrac{1}{x}\right)\right)\, dx \\
&= \pi \int_1^3 \left(x^2+1\right)\, dx \\
&= \pi \left(\tfrac{1}{3}x^3 + x\right)\Big|_{x=1}^{3} \\
&= \pi\left(\tfrac{1}{3}\left(3^3 - 1^3\right) + (3-1)\right) \\
&= \boxed{\frac{32\pi}{3}}.
\end{aligned}$$

11.1: Sequences. Determine whether the sequence converges or diverges. If it converges, find the limit.

(1) $a_n = \dfrac{(\ln n)^2}{\sqrt{n}}$.

Solution: This sequence is related to the function $f(x) = \frac{(\ln x)^2}{\sqrt{x}}$. Does this function have a limit as $x \to \infty$? "Plugging in" $x = \infty$ gives us the indeterminate form $\frac{\infty}{\infty}$, so we can apply L'Hôpital's rule. We find that

$$\lim_{x\to\infty} \frac{(\ln x)^2}{\sqrt{x}} = \lim_{x\to\infty} \frac{2\ln x \cdot \frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{4\ln x}{\sqrt{x}}.$$

Again, we have the indeterminate form $\frac{\infty}{\infty}$, so applying L'Hôpital's rule again, we find that

$$\lim_{x\to\infty} \frac{4\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{4}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{8}{\sqrt{x}}.$$

This last limit is equal to 0. As x becomes very large, $\frac{8}{\sqrt{x}}$ becomes very small. Thus, the limit of the sequence $a_n = f(n)$ is also $\boxed{0}$.

(2) $a_n = \arctan(\ln n)$

Solution: This sequence is related to the function $f(x) = \arctan(\ln x)$. As x becomes very large, $\ln x$ becomes very large as well. As the input to arctan becomes very large, the output becomes close to $\frac{\pi}{2}$. Thus,

$$\lim_{x\to\infty} \arctan(\ln x) = \frac{\pi}{2}.$$

We conclude that the limit of the sequence $a_n = f(n)$ is also equal to $\boxed{\dfrac{\pi}{2}}$.

(3) $a_n = \ln(2n+1) - \ln(n+1)$

Solution: Both terms become large as n becomes large. The difference of two large numbers could be anything, so that's not helpful.

Using log rules, we see that

$$a_n = \ln\left(\frac{2n+1}{n+1}\right).$$

The sequence $\frac{2n+1}{n+1}$ has a limit of 2 as $n \to \infty$. There are a couple ways of seeing this. One is to divide the numerator and denominator by n, giving us $\frac{2+\frac{1}{n}}{1+\frac{1}{n}}$. As $n \to \infty$, the $\frac{1}{n}$ terms shrink to zero, leaving us with $\frac{2}{1} = 2$. Another way is to write $\frac{2n+1}{n+1}$ as $2 - \frac{1}{n+1}$. As $n \to \infty$, the $\frac{1}{n+1}$ term shrinks to zero, leaving us with just 2. Yet another way is to consider the function $f(x) = \frac{2x+1}{x+1}$ and compute its limit as $x \to \infty$.

In any case, once we know that the limit of $\frac{2n+1}{n+1}$ is 2, using the fact that ln is continuous at 2, we know that the limit of a_n is $\boxed{\ln 2}$.

(4) $a_n = n \ln\left(1 + \frac{17}{n}\right)$

Solution: This sequence is related to the function $f(x) = x \ln\left(1 + \frac{17}{x}\right)$. "Plugging in $x = \infty$ gives us $\infty \cdot 0$. A very large number times a tiny number could be anything, so that's not helpful.

To make progress, we should use L'Hôpital's rule, and to do that we must write the expression as a fraction. We write

$$f(x) = \frac{\ln\left(1 + \frac{17}{x}\right)}{\frac{1}{x}}.$$

Now, we have the indeterminate form $\frac{0}{0}$. Using L'Hôpital's rule,

$$\lim_{x\to\infty} \frac{\ln\left(1 + \frac{17}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{17}{x}} \cdot \left(-\frac{17}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{17}{1 + \frac{17}{x}}.$$

This limit we can evaluate. As x becomes very large, $\frac{17}{x}$ goes to zero, so

$$\lim_{x\to\infty} \frac{17}{1 + \frac{17}{x}} = \frac{17}{1+0} = 17.$$

Thus, the limit of the sequence $a_n = f(n)$ is also $\boxed{17}$.

(5) $a_n = \left(1 + \frac{17}{n}\right)^n$

Solution: This formula appears to have some connection to the formula in the previous problem. Investigating, we eventually discover that

$$\ln(a_n) = n \ln\left(1 + \frac{17}{n}\right).$$

Equivalently,

$$a_n = e^{n \ln\left(1+\frac{17}{n}\right)}.$$

We found that the limit of $n \ln\left(1 + \frac{17}{n}\right)$ is 17, and the exponential function is continuous everywhere, so the limit of a_n is $\boxed{e^{17}}$.

(6) $a_n = n \sin\left(\frac{2}{n}\right)$

Solution: This sequence is related to the function $f(x) = x \sin\left(\frac{2}{x}\right)$. "Plugging in" $x = \infty$ gives us $\infty \cdot \sin(0) = \infty \cdot 0$. A large number times a small number could be anything, so we need to apply L'Hôpital's rule. To do so, we must first convert the expression into a fraction. We find that

$$f(x) = \frac{\sin\left(\frac{2}{x}\right)}{\frac{1}{x}}.$$

Now, we have an indeterminate form $\frac{0}{0}$. Applying L'Hôpital's rule, we find that

$$\lim_{x\to\infty} \frac{\sin\left(\frac{2}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\cos\left(\frac{2}{x}\right)\cdot\left(-\frac{2}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} 2\cos\left(\frac{2}{x}\right).$$

This is a limit we can evaluate. As x becomes very large, $\frac{2}{x}$ shrinks to zero, and cosine is continuous at zero, so

$$\lim_{x\to\infty} 2\cos\left(\frac{2}{x}\right) = 2\cos(0) = 2.$$

Therefore, the limit of the sequence $a_n = f(n)$ is also $\boxed{2}$.

(7) $a_n = n - \sqrt{n+1}\sqrt{n+5}$

Solution: "Plugging in" $n = \infty$ gives us $\infty - \infty$. That's not helpful, because the difference of two very large numbers could be anything.

This problem is tricky, and there are many successful approaches. One approach we've seen that works well for sums or differences involving square roots is to multiply and divide by the conjugate $n + \sqrt{n+1}\sqrt{n+5}$ and use the difference of squares formula. We obtain

$$\begin{aligned} a_n &= \left(n - \sqrt{n+1}\sqrt{n+5}\right)\cdot\frac{n+\sqrt{n+1}\sqrt{n+5}}{n+\sqrt{n+1}\sqrt{n+5}} \\ &= \frac{n^2-(n+1)(n+5)}{n+\sqrt{n+1}\sqrt{n+5}} \\ &= \frac{-6n-5}{n+\sqrt{n+1}\sqrt{n+5}}. \end{aligned}$$

At this point, both the numerator and the denominator have "degree" one, so we can understand the ratio by dividing both the numerator and denominator by n. We obtain

$$\begin{aligned} a_n = \frac{-6n-5}{n+\sqrt{n+1}\sqrt{n+5}} &= \frac{-6n-5}{n+\sqrt{n+1}\sqrt{n+5}}\cdot\frac{\frac{1}{n}}{\frac{1}{n}} \\ &= \frac{-6-\frac{5}{n}}{1+\frac{\sqrt{n+1}\sqrt{n+5}}{\sqrt{n}\sqrt{n}}} \\ &= \frac{-6-\frac{5}{n}}{1+\sqrt{1+\frac{1}{n}}\sqrt{1+\frac{5}{n}}}. \end{aligned}$$

As n becomes very large, $\frac{5}{n}$ and $\frac{1}{n}$ shrink to zero, so

$$\lim_{n\to\infty} a_n = \frac{-6-0}{1+\sqrt{1+0}\sqrt{1+0}} = -\frac{6}{2} = \boxed{-3}.$$

Note that this sequence comes from the function $f(x) = x - \sqrt{x+1}\sqrt{x+5}$, and we could use the same technique to compute $\lim_{x\to\infty} f(x)$.