SOLUTIONS TO

MATH 132 MIDTERM EXAM 2

This exam consists of 10 multiple choice (machine-graded) problems, worth 3 points each (for a total of 30 points), and 2 pages of written (hand-graded) problems, worth a total of 20 points. No 3x5 cards or calculators are allowed.

You may need the half-angle formulas

$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$, $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\sin 2\theta$.

PART I: MULTIPLE CHOICE PROBLEMS

You will need a pencil to mark your card. If you do not have one, please ask your proctor. Write your **ID number** (not your SS number) on the six blank lines on the top of your answer card, using one blank for each digit. **Shade in the corresponding boxes below.** Also **print your name at the top of your card.**

(1) Compute the definite integral $\int_1^2 \frac{dx}{x^2}$.
(A) $\frac{1}{2}$
(B) $\frac{3}{4}$
(C) $\frac{7}{8}$
(D) $\frac{21}{8}$

$$= \int_1^2 x^{-2}\,dx$$
$$= \left[\frac{x^{-1}}{-1}\right]_1^2 = \left[x^{-1}\right]_2^1$$
$$= 1 - \frac{1}{2}$$
$$= \frac{1}{2}.$$

(2) Find $G'(\sqrt{2\pi})$ if $G(x) := \int_1^{x^2} \sqrt{2+\sin(t)}\,dt$.
(A) $2\sqrt{\pi}$
(B) 2π
(C) $4\sqrt{\pi}$
(D) 4π

$$G'(x) = \frac{d}{dx}x^2 \cdot \sqrt{2+\sin(x^2)} = 2x\cdot\sqrt{2+\sin(x^2)}$$
$$G'(\sqrt{2\pi}) = 2\sqrt{2\pi}\cdot\sqrt{2+\cancel{\sin(2\pi)}} \to 0$$
$$= 2\sqrt{2\pi}\,\sqrt{2}$$
$$= 4\sqrt{\pi}$$

(3) Find $\int_1^{\sqrt{6}} 3x\sqrt{x^2+3}dx$. $= \frac{3}{2}\int_4^9 \sqrt{u}\,du = \frac{3}{2}\cdot\frac{u^{3/2}}{3/2}\Big|_4^9$

(A) 11
(B) 15
(C) 19
(D) 23

$u = x^2+3$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$x=1 \to u=4$

$x=\sqrt{6} \to u=9$

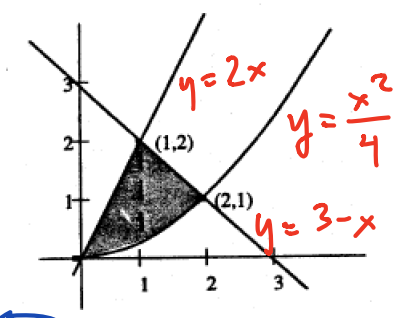
$= (\sqrt{u})^3\Big|_4^9$

$= 3^3 - 2^3$

$= 27 - 8$

$= 19$

(4) Determine the area of the region bounded by the curves $y=2x$, $y=x^2/4$ and $y=3-x$.



(A) $\frac{11}{6}$
(B) $\frac{13}{6}$
(C) $\frac{17}{6}$
(D) $\frac{23}{6}$

$$\int_0^1 \left(2x - \frac{x^2}{4}\right)dx + \int_1^2 \left(3-x-\frac{x^2}{4}\right)dx$$

$$= \left[x^2 - \frac{x^3}{12}\right]_0^1 + \left[3x - \frac{x^2}{2} - \frac{x^3}{12}\right]_1^2$$

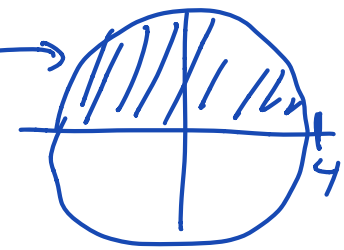
$$= 1 - \cancel{\frac{1}{12}} + 6 - 2 - \frac{8}{12} - 3 + \frac{1}{2} + \cancel{\frac{1}{12}}$$

$$= \frac{5}{2} - \frac{2}{3} = \frac{15-4}{6} = \frac{11}{6}$$

(5) Find the average value of $f(x) = \sqrt{16 - x^2}$ on $[-4, 4]$.

(A) 0
(B) π
(C) 2π
(D) 4π

$$f_{avg} = \frac{1}{8}\int_{-4}^{4} \sqrt{4^2 - x^2}\, dx$$



$$= \frac{1}{8} \cdot \frac{1}{2}\pi(4)^2$$

$$= \frac{1}{16}\pi \cdot 16 = \pi$$

(6) Compute the definite integral $\int_0^1 t^2 e^t dt$.

(A) $e - 1$
(B) $2e - 1$
(C) $e - 2$
(D) $2e - 2$

$$= t^2 e^t \Big|_0^1 - 2\int_0^1 t e^t\, dt$$

$$\begin{pmatrix} u = t^2 & dv = e^t dt \\ du = 2t\,dt & v = e^t \end{pmatrix}$$

$$= e - 2\Big(t e^t \Big|_0^1 - \underbrace{\int_0^1 e^t\, dt}_{e^t\big|_0^1}\Big)$$

$$\begin{pmatrix} u = t & dv = e^t dt \\ du = dt & v = e^t \end{pmatrix}$$

$$= e - 2(\cancel{e} - (\cancel{e} - 1))$$

$$= e - 2$$

(7) Find the value of $\int_0^{\frac{\pi}{2}} \sin^2 x \cos^3 x \, dx$. $= \int_0^{\pi/2} \sin^2 x (1-\sin^2 x) \cos x \, dx$

(A) $\frac{1}{5}$
(B) $\frac{1}{15}$
(C) $\frac{2}{5}$
(D) $\frac{2}{15}$

$= \int_0^1 t^2(1-t^2)\, dt$

$t = \sin x$
$dt = \cos x \, dx$

$= \int_0^1 (t^2 - t^4)\, dt$

$= \left[\frac{t^3}{3} - \frac{t^5}{5}\right]_0^1$

$= \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$

(8) Compute the definite integral $\int_2^3 \frac{dx}{x^2-6x+5}$. $= \frac{1}{4}\int_2^3 \frac{dx}{x-5} - \frac{1}{4}\int_2^3 \frac{dx}{x-1}$

(A) $-3\ln 2$
(B) $-\frac{1}{3}\ln 2$
(C) $-4\ln 3$
(D) $-\frac{1}{4}\ln 3$

$\frac{1}{x^2-6x+5} = \frac{1}{(x-5)(x-1)} = \frac{A}{x-5} + \frac{B}{x-1} = \frac{(A+B)x + (-5B-A)}{(x-5)(x-1)}$

$A+B=0$
$-A-5B=1$

$-4B = 1 \Rightarrow B = -\frac{1}{4}, \; A = \frac{1}{4}.$

$= \frac{1}{4}\ln|x-5|\Big|_2^3 - \frac{1}{4}\ln|x-1|\Big|_2^3$

$= \frac{1}{4}\{\cancel{\ln 2} - \ln 3 - \cancel{\ln 2} + \cancel{\ln 1} \to 0\}$

$= -\frac{1}{4}\ln 3$

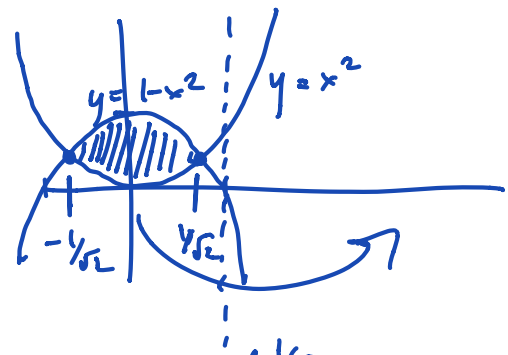
(9) Which of the following is *not* an integral computing the volume of the solid obtained by revolving the region between $y = 1 - x^2$ and $y = x^2$ about the line $x = 1$?

(A) $2\pi \int_{-1}^{1} (1 - 2x^2)(1 - x)dx$

(B) $2\pi \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} (1 - 2x^2)(1 - x)dx$ ✓

(C) $2\pi \int_{0}^{1/2} 4\sqrt{y}dy$

(D) $2\pi \int_{0}^{1/2} 4\sqrt{1 - y}dy$



shells: $\int_{-1/\sqrt{2}}^{1/\sqrt{2}} 2\pi (1-2x^2)(1-x)\, dx$

washers: $2\int_0^{1/2} \pi\left((1+\sqrt{y})^2 - (1-\sqrt{y})^2\right) dx$

$= 2\pi \int_0^{1/2} 4\sqrt{y}\, dy$

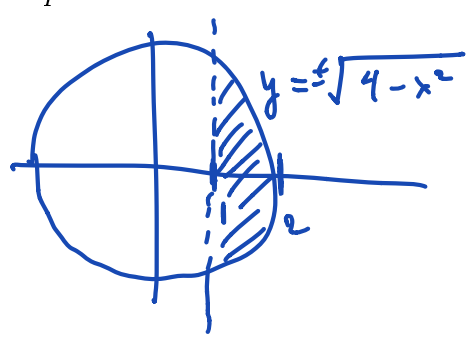
(10) Find the area of the *portion* of the circle of radius 2 (centered at $(0, 0)$) *lying to the right of the line* $x = 1$.

(A) $2\sqrt{3} - \frac{\pi}{3}$

(B) $\frac{4\pi}{3} - \sqrt{3}$

(C) $4\sqrt{3} - 2\pi$

(D) $\frac{5\pi}{3} - 2\sqrt{3}$



$A = \int_1^2 2\sqrt{4-x^2}\, dx$

$= \int_{\pi/6}^{\pi/2} 2\underbrace{\sqrt{4-4\sin^2\theta}}_{2\cos\theta}\, 2\cos\theta\, d\theta$

$x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$x = 1 \to \frac{1}{2} = \sin\theta \to \theta = \frac{\pi}{6}$

$x = 2 \to 1 = \sin\theta \to \theta = \frac{\pi}{2}$

$= 8\int_{\pi/6}^{\pi/2} \cos^2\theta\, d\theta$

$= 4\int_{\pi/6}^{\pi/2} (1 + \cos 2\theta)\, d\theta$

$= 4\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\pi/6}^{\pi/2}$

$= 4\left(\frac{\pi}{2} - \frac{\pi}{6}\right) + 2\left(\cancel{\sin\pi}^{\,0} - \sin\pi/3\right) = \frac{4\pi}{3} - \sqrt{3}$

$\sqrt{3}/2$

Part II: Hand-graded problems

This part has two pages. Show all the work you want graded for each problem in the space provided. Please **print your name at the top of each page.**

(1) The new SpaceX rocket consumes its fuel at a constant rate of 200 kg/s, deriving from this a constant upward thrust of 600,000 N. Assuming an initial mass (including fuel) of 30,000 kg at liftoff, find the height of the rocket after 100 seconds. [Recall $N = kg \cdot m/s^2$, and Newton's second law $F = ma$.]

(a) [4 points] Determine the upward acceleration as a function of time. (Use $g = \cancel{10\,m/s^2}$, and don't worry about fuel running out; express your answer in km/s^2.) $\frac{1}{100}\,km/s^2$

Use $F = ma$ for the part of accel. derived from thrust: $\frac{600000\,N}{(30{,}000\,kg - 200\,kg/s \cdot t)} = \frac{3000}{150-t}$ (in m/s^2)

$\Rightarrow \frac{3}{150-t}$ (in km/s^2).

$$a(t) = \frac{3}{150-t} - \frac{1}{100} \quad (\text{in } km/s^2)$$

(b) [3 points] Find the upward velocity $v(t)$ (again, valid until fuel runs out), starting from rest at $t = 0$.

$$v(t) = \int_0^t a(u)\,du = \left[-3\ln|150-u| - \frac{u}{100}\right]_0^t$$

$$= 3\ln 150 - 3\ln(150-t) - \frac{t}{100}$$

(c) [4 points] Use integration by parts to find the height $h(t)$ (again, starting at $h(0) = 0$), and solve the problem.

First note $-\int \ln(150-x)\,dx = \int \ln s\,ds$

$$\begin{pmatrix} s = 150 - x \\ ds = -dx \end{pmatrix}$$

$$= s\ln s - \int \frac{1}{\cancel{s}}\cancel{s}\,ds = s\ln s - s + C.$$

$$\begin{pmatrix} u = \ln s & dv = ds \\ du = \frac{1}{s}ds & v = s \end{pmatrix}$$

$$= (150-x)\ln(150-x) + x + C$$

$$\int_0^t v(x)\,dx = \left[3\ln(150)x + 3(150-x)\ln(150-x) + 3x - \frac{x^2}{200}\right]_0^t$$

$$= 3\ln(150)t + 3(150-t)\ln(150-t) + 3t - \frac{t^2}{200} - 3\cdot 150\ln(150)$$

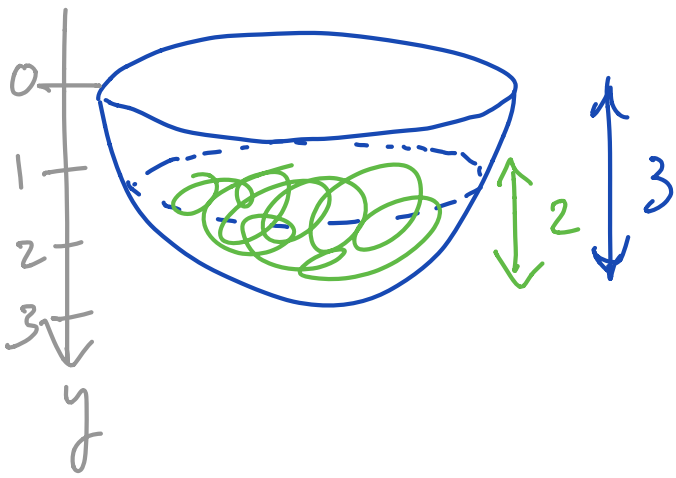
$$h(100) = 300\ln 150 + 150\ln 50 + 300 - 50 - 450\ln 150$$

$$= 150(\ln 50 - \ln 150) + 250$$

$$= -150\ln 3 + 250 \quad (\text{in km})$$

$$\approx 85 \text{ km} \quad (\text{using } \ln 3 \approx 1.1)$$

(2) (a) [5 points] Your backyard swimming pool is in the shape of a hemisphere of radius (and thus depth) 3 meters. The top meter is empty and the bottom 2 meters are filled with slimy algae. Compute the volume (in m^3) of the algae.



$$V = \int_1^3 A(y)\,dy$$

$$= \int_1^3 \pi\left(\sqrt{3^2 - y^2}\right)^2 dy$$

$$= \pi \int_1^3 (9 - y^2)\,dy$$

$$= \pi \left[9y - y^3/3\right]_1^3$$

$$= \pi\left\{(27 - 9) - (9 - 1/3)\right\}$$

$$= \pi\left(9 + \tfrac{1}{3}\right) = \frac{28\pi}{3}.$$

(b) [5 points] If the algae slime weighs $1000\,kg/m^3$, determine the amount of work (in Joules) required to pump it all out (i.e. to ground level).

$$W = g\int_1^3 1000 \cdot A(y)\, y\, dy = \int_1^3 10^4 \pi (9 - y^2)\, y\, dy$$

$$= 10^4 \pi \int_1^3 (9y - y^9)\,dy = 10^4 \pi \left[\frac{9}{2}y^2 - \frac{y^4}{4}\right]_1^3$$

$$= 10^4 \pi \left\{\underbrace{\left(\frac{81}{2} - \frac{81}{4}\right)}_{81/4} - \underbrace{\left(\frac{9}{2} - \frac{1}{4}\right)}_{17/4}\right\}$$

$$= 10^4\pi \cdot \frac{81-17}{4}$$

$$= 10000\pi \cdot \frac{\overset{16}{\cancel{64}}}{\cancel{4}}$$

$$= 160000\pi \quad J.$$