SOLUTIONS TO

MATH 132 MIDTERM EXAM 2

This exam consists of 10 multiple choice (machine-graded) problems, worth 3 points each (for a total of 30 points), and 2 pages of written (hand-graded) problems, worth a total of 20 points. No 3x5 cards or calculators are allowed.

You may need the half-angle formulas

\[
\cos^2 \theta = \frac{1}{2} + \frac{1}{2} \cos 2\theta, \quad \sin^2 \theta = \frac{1}{2} - \frac{1}{2} \sin 2\theta.
\]

Part I: Multiple choice problems

You will need a pencil to mark your card. If you do not have one, please ask your proctor. Write your ID number (not your SS number) on the six blank lines on the top of your answer card, using one blank for each digit. Shade in the corresponding boxes below. Also print your name at the top of your card.

1. Compute the definite integral \( \int_1^2 \frac{dx}{x^2} \).
   \[ \int_1^2 \frac{dx}{x^2} = \left[ \frac{x^{-1}}{-1} \right]_1^2 = \left[ x^{-1} \right]_1^2 = 1 - \frac{1}{2} = \frac{1}{2}. \]

2. Find \( G'(\sqrt{2\pi}) \) if \( G(x) := \int_1^x \sqrt{2 + \sin(t)} \, dt \).
   \[ G'(x) = \frac{d}{dx} \sqrt{2 + \sin(k^2)} = 2x \cdot \sqrt{2 + \sin(k^2)} \]
   \[ G'(\sqrt{2\pi}) = 2 \sqrt{2\pi} \cdot \sqrt{2 + \sin(k^2)} = 2 \sqrt{2\pi} \cdot \sqrt{2} = 4\sqrt{\pi} \]
(3) Find \( \int_1^{\sqrt{6}} 3x^3 + 3 dx \).
(A) 11
(B) 15
(C) 19
(D) 23

\[
\int_1^{\sqrt{6}} 3x^3 + 3 dx = \frac{3}{2} \int_4^9 \sqrt{u} \, du = \frac{3}{2} \cdot \left. \frac{u^{3/2}}{\frac{3}{2}} \right|_4^9 = (\sqrt{u})^3 \bigg|_4^9 = 3^3 - 2^3 = 27 - 8 = 19
\]

(4) Determine the area of the region bounded by the curves \( y = 2x \), \( y = \frac{x^2}{4} \) and \( y = 3 - x \).

\[
\int_0^1 \left( 2x - \frac{x^2}{4} \right) \, dx + \int_1^2 \left( 3 - x - \frac{x^2}{4} \right) \, dx = \int \left[ x^2 - \frac{x^3}{12} \right]_0^1 + \left[ 3x - \frac{x^2}{2} - \frac{x^3}{12} \right]_1^2 \left. \right| \, dx = 1 - \frac{1}{2} + 6 - 2 - \frac{8}{12} - 3 + \frac{1}{2} + \frac{1}{2} = \frac{5}{2} - \frac{2}{3} = \frac{15 - 4}{6} = \frac{11}{6}
\]
(5) Find the average value of \( f(x) = \sqrt{16 - x^2} \) on \([-4, 4]\).

(A) 0
(B) \( \pi \)
(C) 2\( \pi \)
(D) 4\( \pi \)

\[
\frac{1}{\pi} \int_{-4}^{4} \sqrt{16 - x^2} \, dx = \frac{1}{8} \left[ \frac{1}{2} \pi (4)^2 \right] = \frac{1}{16} \cdot 16 = \pi
\]

(6) Compute the definite integral \( \int_{0}^{1} t^2 e^t \, dt \).

(A) \( e - 1 \)
(B) \( 2e - 1 \)
(C) \( e - 2 \)
(D) \( 2e - 2 \)

\[
\int_{0}^{1} t^2 e^t \, dt = t^2 e^t \bigg|_{0}^{1} - 2 \int_{0}^{1} te^t \, dt
\]

\[
(u = t^2, \quad dv = e^t \, dt)
\]

\[
(du = 2t \, dt, \quad v = e^t)
\]

\[
= e - 2 \left( te^t \bigg|_{0}^{1} - \int_{0}^{1} e^t \, dt \right)
\]

\[
(u = t, \quad dv = e^t \, dt)
\]

\[
(du = dt, \quad v = e^t)
\]

\[
= e - 2 \left( e - (e^1 - 1) \right)
\]

\[
= e - 2
\]
(7) Find the value of \( \int_{0}^{\pi/2} \sin^2 x \cos^3 x \, dx \).

\[
\begin{align*}
(A) \quad & \frac{1}{5} \\
(B) \quad & \frac{1}{15} \\
(C) \quad & \frac{2}{15} \\
(D) \quad & \frac{7}{15}
\end{align*}
\]

(8) Compute the definite integral \( \int_{2}^{3} \frac{dx}{x^2 - 6x + 5} \).

\[
\begin{align*}
(A) \quad & -3 \ln 2 \\
(B) \quad & -\frac{1}{3} \ln 2 \\
(C) \quad & -4 \ln 3 \\
(D) \quad & -\frac{4}{3} \ln 3
\end{align*}
\]
(9) Which of the following is not an integral computing the volume of the solid obtained by revolving the region between \( y = 1 - x^2 \) and \( y = x^2 \) about the line \( x = 1 \)?

(A) \( 2\pi \int_{-1}^{1} (1 - 2x^2)(1 - x) \, dx \)

(B) \( 2\pi \int_{\frac{1}{\sqrt{2}}}^{\frac{3}{\sqrt{2}}} (1 - 2x^2)(1 - x) \, dx \)

(C) \( 2\pi \int_{0}^{4\sqrt{y}} \)

(D) \( 2\pi \int_{0}^{4\sqrt{y}} \)

\[
\begin{align*}
\text{Shells:} & \quad \int_{-1/\sqrt{2}}^{1/\sqrt{2}} 2\pi (1 - 2x^2)(1 - x) \, dx \\
\text{Washers:} & \quad 2 \int_{\frac{1}{\sqrt{2}}}^{\frac{3}{\sqrt{2}}} \pi \left( (1 + \sqrt{y})^2 - (1 - \sqrt{y})^2 \right) \, dx \\
& = 2\pi \int_{0}^{4\sqrt{y}} y \, dy
\end{align*}
\]

(10) Find the area of the portion of the circle of radius 2 (centered at (0, 0)) lying to the right of the line \( x = 1 \),

(A) \( 2\sqrt{3} - \frac{\pi}{3} \)

(B) \( \frac{4\pi}{3} - \sqrt{3} \)

(C) \( 4\sqrt{3} - 2\pi \)

(D) \( \frac{5\pi}{3} - 2\sqrt{3} \)

\[
\begin{align*}
A & = \int_{1}^{2} 2\sqrt{4 - x^2} \, dx \\
& = \int_{\pi/6}^{\pi/3} 2\sqrt{4 - 4\sin^2 \theta} \cdot 2\cos \theta \, d\theta \\
& = \int_{\pi/6}^{\pi/3} 2\sqrt{1 - \sin^2 \theta} \cdot 2\cos \theta \, d\theta \\
& = \left. 2\cos \theta \right|_{\pi/6}^{\pi/3} \\
& = \left( 2\cos \frac{\pi}{3} - 2\cos \frac{\pi}{6} \right) \\
& = 2\left( \frac{1}{2} - \frac{\sqrt{3}}{2} \right) \\
& = \frac{\sqrt{3}}{2} - \sqrt{3}
\end{align*}
\]
PART II: HAND-GRADED PROBLEMS

This part has two pages. Show all the work you want graded for each problem in the space provided. Please print your name at the top of each page.

(1) The new SpaceX rocket consumes its fuel at a constant rate of 200 kg/s, deriving from this a constant upward thrust of 600,000 N. Assuming an initial mass (including fuel) of 30,000 kg at liftoff, find the height of the rocket after 100 seconds. [Recall $N = \text{kg} \cdot \text{m/s}^2$, and Newton's second law $F = ma$.]

(a) [4 points] Determine the upward acceleration as a function of time. (Use $g = \frac{3}{100}$ km/s$^2$, and don't worry about fuel running out; express your answer in km/s$^2$.)

$$F = ma$$

Use $F = ma$ for the part of accel. derived from thrust: $\frac{600000 \text{ N}}{60,000 \text{ kg} - 200 \text{ kg/s} \cdot t} = \frac{3000}{150 - t}$ (in km/s$^2$)

$$= \frac{3}{150 - t} \text{ (in km/s}^2\text{).}$$

$$a(t) = \frac{3}{150 - t} - \frac{1}{100} \text{ (in km/s}^2\text{).}$$

(b) [3 points] Find the upward velocity $v(t)$ (again, valid until fuel runs out), starting from rest at $t = 0$.

$$V(t) = \int_0^t a(u) \, du = \left[-3 \ln|150-u| - \frac{u}{100}\right]_0^t$$

$$= 3 \ln 150 - 3 \ln(150-t) - \frac{t}{100}$$
(c) [4 points] Use integration by parts to find the height \( h(t) \) (again, starting at \( h(0) = 0 \)), and solve the problem.

\[
\text{First note} \quad \int \ln (150 - x) \, dx = \int \ln s \, ds \\
\begin{align*}
&= s \ln s - \int \frac{1}{s} \, ds = s \ln s - s + C \\
&\quad \left( u = \ln s, \quad dv = ds \right) \\
&\quad \left( du = \frac{1}{s} \, ds, \quad v = s \right)
\end{align*}
\]

\[
\int_0^t v(t) \, dx = \left[ 3 \ln(150) + 3(150 - x) \ln(150 - x) + 3x - \frac{x^2}{200} \right]_0^t
\]

\[
= 3 \ln(150) t + 3(150 - t) \ln(150 - t) + 3t - \frac{t^2}{200} - 3 \cdot 150 \ln(150)
\]

\[
h(100) = 300 \ln 150 + 150 \ln 50 + 300 - 50 - 450 \ln 150
\]

\[
= 150 \left( \ln 50 - \ln 150 \right) + 250
\]

\[
= -150 \ln 3 + 250 \text{ (in km)}
\]

\[
\approx 85 \text{ km} \text{ (using } \ln 3 \approx 1.1)\]
(2) (a) [5 points] Your backyard swimming pool is in the shape of a hemisphere of radius (and thus depth) 3 meters. The top meter is empty and the bottom 2 meters are filled with slimy algae. Compute the volume (in m$^3$) of the algae.

\[
V = \int_1^3 A(y) \, dy \\
= \int_1^3 \frac{3}{\pi} (\sqrt{3^2 - y^2})^2 \, dy \\
= \pi \int_1^3 (9 - y^2) \, dy \\
= \pi \left[ \frac{9y - y^3}{3} \right]_1^3 \\
= \pi \left\{ \left[ (27 - 9) - (9 - \frac{1}{3}) \right] \right\} \\
= \frac{28\pi}{3} \, m^3
\]

(b) [5 points] If the algae slime weighs 1000 kg/m$^3$, determine the amount of work (in Joules) required to pump it all out (i.e. to ground level).

\[
W = \frac{g}{9} \int_1^3 1000 A(y) y \, dy = \int_1^3 \frac{9}{\pi} (9 - y^2) y \, dy \\
= 10\frac{g}{9} \int_1^3 (9y - y^3) \, dy = 10\frac{g}{9} \left[ \frac{9y^2}{2} - \frac{y^4}{4} \right]_1^3 \\
= 10\frac{g}{9} \left\{ \left( \frac{81}{2} - \frac{81}{4} \right) - \left( \frac{9}{2} - \frac{1}{4} \right) \right\}
\]
\[ = 10^4 \pi \cdot \frac{81 - 17}{4} \]

\[ = 10000 \pi \cdot \frac{64}{4} \]

\[ = 160000 \pi \ J. \]