Lecture 10: Volumes of solids

How would you find the volume of a bowling ball?

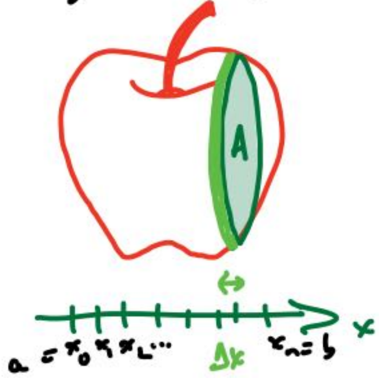
Applied mathematician: $\frac{4}{3}\pi\left(\frac{\text{measured circumference}}{2\pi}\right)^3$

Physicist: drop it in a gallon of water & measure displacement!

Engineer: write down the serial number & look it up?

Pure mathematician: write down Riemann sum, ... haha.

Let's say this pure mathematician is making her breakfast



... which of course involves an apple and a mandoline. Since we're counting calories, we'll need to know the volume of each slice, $\Delta V \approx \Delta x \cdot A$.

The area of the cross section depends on x, so we write it as a function $A(x)$. Adding up the approximate volumes of n slices of equal width gives

$$V_{\text{Apple}} = \sum_{i=1}^{n} (\Delta V)_i \approx \Delta x \sum_{i=1}^{n} A(x_i).$$

Taking the limit as $\Delta x \to 0$ $(n \to \infty)$ gives

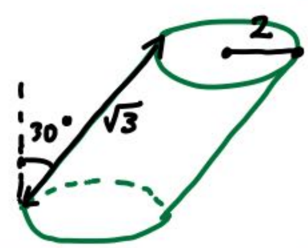
$$\left(V_{\text{Apple}} =\right) \lim_{n\to\infty} \Delta x \sum_{i=1}^{n} A(x_i) = \int_a^b A(x)\, dx.$$

This is a simple formula but already illustrates the principle first articulated by Bonaventura Cavalieri in 1635: that every solid is composed of an infinite # of planar cross-sections, and so long as these cross-sections have the same area at each height, the volumes are equal.

So this apple . . . and this one . . .

have the same volume, because the functions $A(x)$ are the same!

$\boxed{\text{Try}}$ Find the volume of the "skew" cylinder



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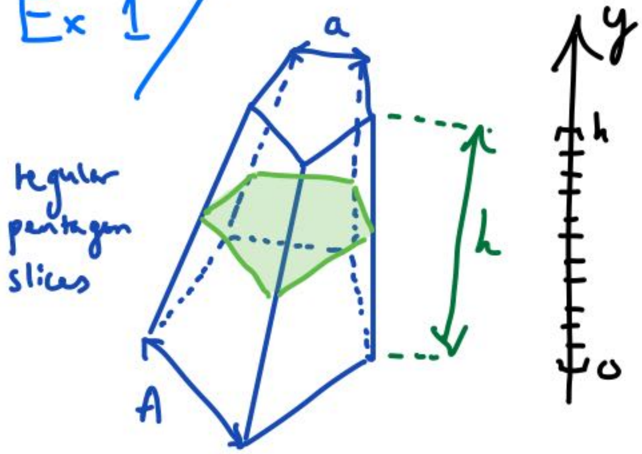
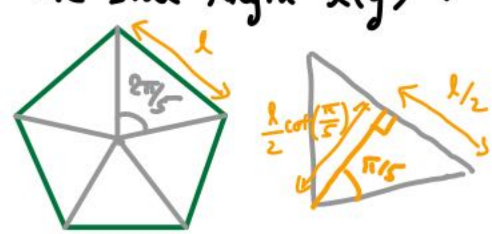
Answer: 6π

Ex 1/



Each slice has an area $A(y)$, which depends on the side length $\ell(y)$:

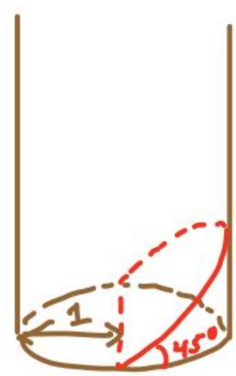


through the formula $A = \frac{5\ell^2}{4}\cot\left(\frac{\pi}{5}\right)$ (why?).
As for $\ell(y)$, this depends linearly on y with $\ell(0) = A$ and $\ell(h) = a$. So $\ell(y) = A + \frac{(a-A)}{h}y$, and
$A(y) = \frac{5}{4}\cot\left(\frac{\pi}{5}\right)\cdot\left(A + \frac{(a-A)}{h}y\right)^2$. To get the volume, we compute

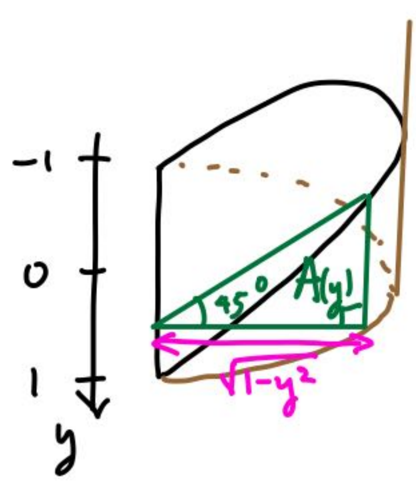
$$V = \int_0^h A(y)\,dy = \frac{5}{4}\cot\left(\frac{\pi}{5}\right)\frac{h}{a-A}\cdot\frac{1}{3}\underbrace{\left[\left(A + \left(\frac{a-A}{h}\right)y\right)^3\right]_0^h}_{a^3 - A^3}$$

$$= \frac{5h}{12}\cot\left(\frac{\pi}{5}\right)\cdot\frac{A^3 - a^3}{A - a} = \frac{5h}{12}\cot\left(\frac{\pi}{5}\right)\left(A^2 + aA + a^2\right).$$

Ex 2 / (Lumberjack problem)



What is the volume of the chunk the lumberjack just chopped out of this tree of 1 ft. radius?



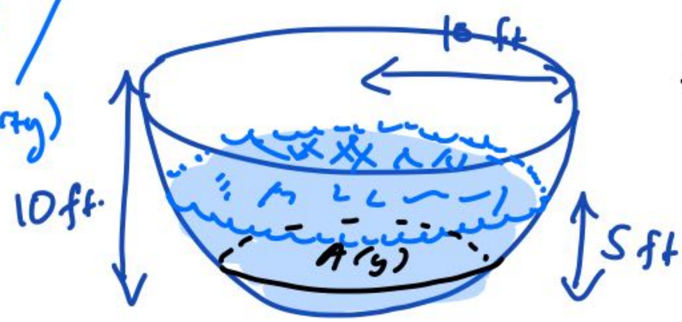
same since it's a 45-45-90 triangle

$$A(y) = \frac{1}{2}\,\text{base}\cdot\text{height}$$
$$= \frac{1}{2}\sqrt{1-y^2}\cdot\sqrt{1-y^2}$$
$$= \frac{1}{2} - \frac{1}{2}y^2$$

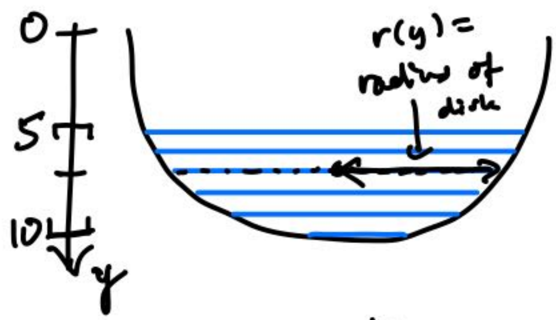
$$V = \int_{-1}^{1} \left(\frac{1}{2} - \frac{1}{2}y^2\right) dy = \left[\frac{1}{2}y - \frac{1}{6}y^3\right]_{-1}^{1}$$

$$= \left(\frac{1}{2} - \frac{1}{6}\right) - \left(-\frac{1}{2} + \frac{1}{6}\right) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3} \text{ ft}^3 .$$

Ex 3 / (pool party)



How much water do we need to fill it up this far?



We have $r(y) = \sqrt{10^2 - y^2}$

$$\Rightarrow A(y) = \pi r(y)^2 = \pi(100 - y^2),$$

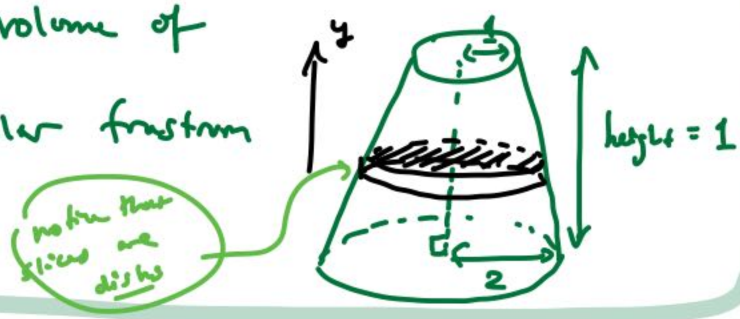
and so the volume is

$$V = \pi \int_{5}^{10} (100 - y^2)\, dy = \pi \left[100y - \frac{y^3}{3}\right]_{5}^{10}$$

$$= \pi\left(\left[1000 - \frac{1000}{3}\right] - \left[500 - \frac{125}{3}\right]\right)$$

$$= \pi\left(\frac{2000}{3} - \frac{1375}{3}\right) = \frac{\pi}{3} \cdot 625 \approx 654.5 \text{ ft}^3 .$$

Try Find the volume of the right circular frustrum displayed:

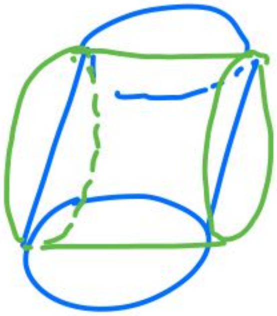


Solution: $y \in [0,1]$, $r(y) = 2 - y$, $A(y) = \pi(2-y)^2 = \pi(y-2)^2$

$$\Rightarrow V = \pi \int_0^1 (y-2)^2 \, dy = \frac{\pi}{3}\left[(y-2)^3\right]_0^1$$

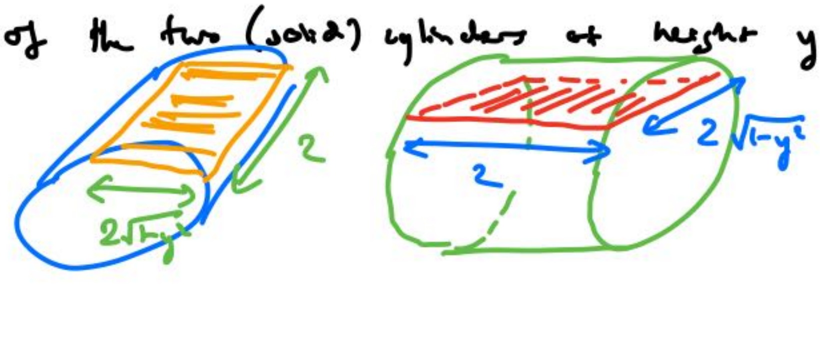
$$= \frac{\pi}{3}\left\{(1-2)^3 - (0-2)^3\right\} = \frac{\pi}{3}\left\{-1+8\right\} = \frac{7\pi}{3}.$$

Ex 4/

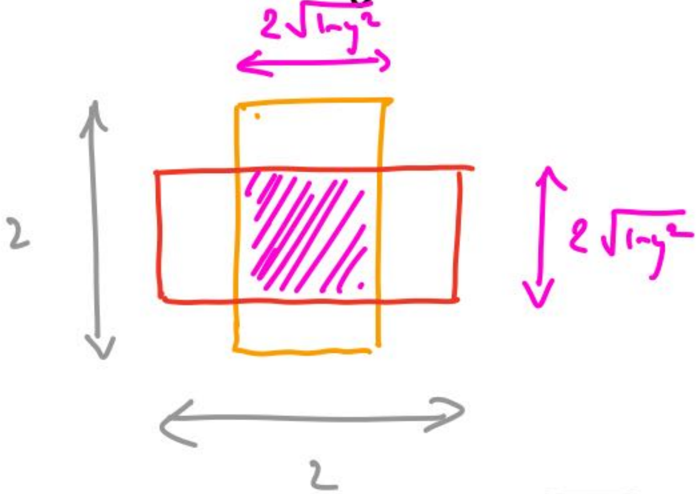


We consider the intersection of two cylinders, meeting at right angles (& having the same centers), both of length 2 and radius 1.

The key point about the intersection is that if we slice it at height y, that's the same as intersecting the slices of the two (solid) cylinders at height y:



That is, we're intersecting



which gives a <u>square</u> of side $2\sqrt{1-y^2} \Longrightarrow$

$$A(y) = \left(2\sqrt{1-y^2}\right)^2 = 4(1-y^2) = 4-4y^2 \Longrightarrow$$

$$V = \int_{-1}^{1} (4-4y^2)\,dy = \left[4y - \frac{4}{3}y^3\right]_{-1}^{1}$$

$$= \left[4 - \frac{4}{3}\right] - \left[-4 + \frac{4}{3}\right] = 2\left[4 - \frac{4}{3}\right] = 2 \cdot \frac{8}{3}$$

$$= \frac{16}{3}.$$

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These examples were, to be honest, mostly hard ones. The problems get easier from here on out, because we will be focusing on volumes of surfaces of revolution, computed by one of three methods:

- disks
- washers

these both fall under the general aegis of the "slicing" we've been doing in this lecture, but the slices are always  or 

- shells

this is totally different and will get its own lecture!