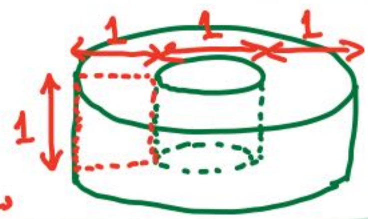
Lecture 11: Slicing up a revolution

Today we'll discuss a special case of computing volumes of solids by slicing: solids of revolution, whose slices are disks or washers.

First, we'll have dessert:

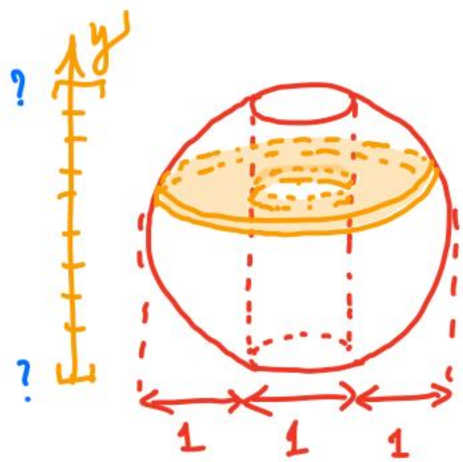
$\boxed{\text{Try}}$ Find the volume of the cake donut!



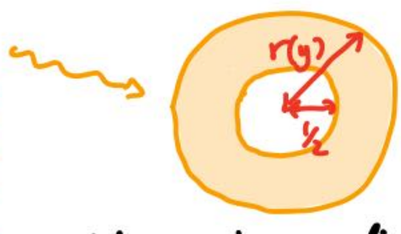
units = inches

Ans.: 2π

Ex 1/ Next, an apple — first, we'll core it...

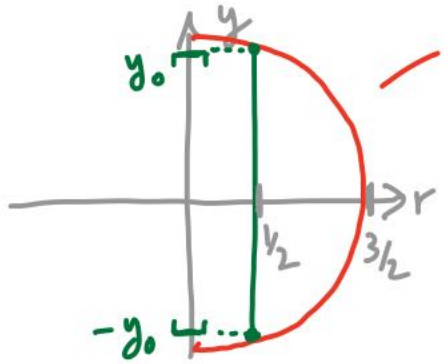


... then slice it:



$A(y) = \pi\, r(y)^2 - \pi\left(\tfrac{1}{2}\right)^2 = \pi\left(r(y)^2 - \tfrac{1}{4}\right)$

We need to find the outer radius $r(y)$ and the limits of integration.



$$r^2+y^2=(3/2)^2$$
$$r(y)=\sqrt{(3/2)^2-y^2}$$

limits of integration: y_0 given by

$$y_0^2+(1/2)^2=(3/2)^2$$
$$y_0^2=\frac{9}{4}-\frac{1}{4}=\frac{8}{4}=2 \Rightarrow y_0=\sqrt{2}$$

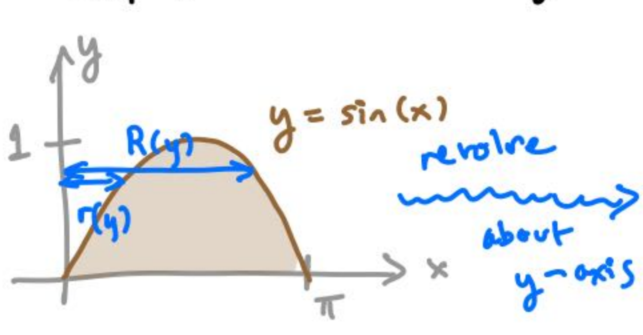
So the volume of the cored apple is

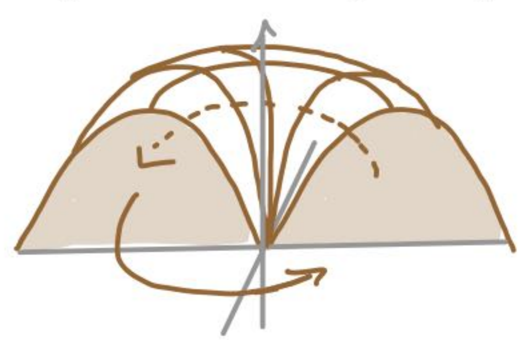
$$V=\int_{-\sqrt{2}}^{\sqrt{2}} A(y)\,dy = 2\int_0^{\sqrt{2}} A(y)\,dy$$
$$=2\pi\int_0^{\sqrt{2}}\left(r(y)^2-1/4\right)dy = 2\pi\int_0^{\sqrt{2}}\left(9/4-y^2-1/4\right)dy$$
$$=2\pi\int_0^{\sqrt{2}}(2-y^2)\,dy = 2\pi\left[2y-\frac{y^3}{3}\right]_0^{\sqrt{2}} = \frac{8\sqrt{2}\pi}{3}.$$
$$\approx 11.8 \text{ in}^3$$

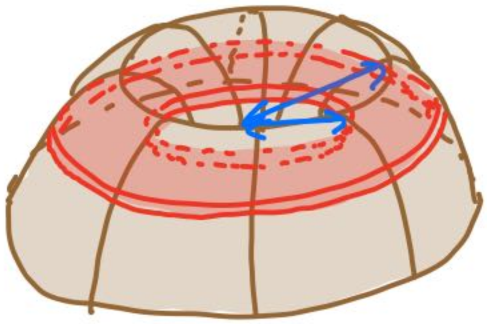
(compare to $\frac{4}{3}\pi\left(\frac{3}{2}\right)^3=\frac{9}{2}\pi \approx 14.1 \text{ in}^3$)

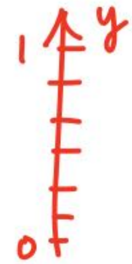
Ex 2/ How about a trigonometric bundt cake?

(See visualization at
http://www.mathdemos.org/mathdemos/washermethod/gallery.)









inner & outer radii:

$r(y) = \arcsin(y)$

$R(y) = \pi - \arcsin(y)$

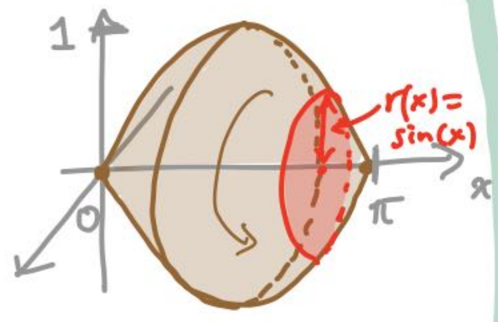
$$V = \int_0^1 A(y)\,dy = \pi\int_0^1 \left(R(y)^2 - r(y)^2\right) dy$$

$$= \pi \int_0^1 \left((\pi - \arcsin(y))^2 - \arcsin(y)^2\right) dy$$

$$= \pi \int_0^1 \left(\pi^2 - 2\pi\arcsin(y)\right) dy \quad \ldots$$

which at the moment is a dead end — we don't know how to antidifferentiate arcsin(y). (Integration by parts will do that job later.)

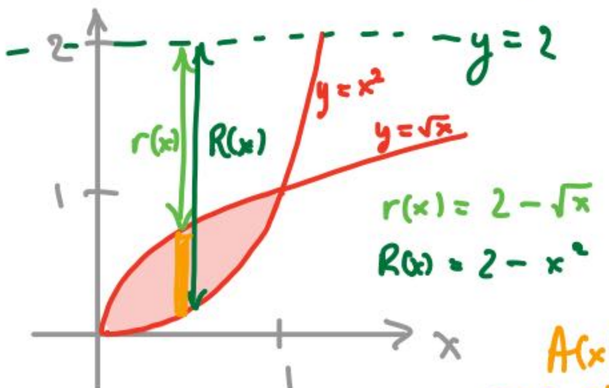
$\boxed{\text{Try}}$ Find the volume of the solid obtained by rotating the region enclosed by $y = \sin(x)$ and the x-axis (from 0 to π) about the x-axis.

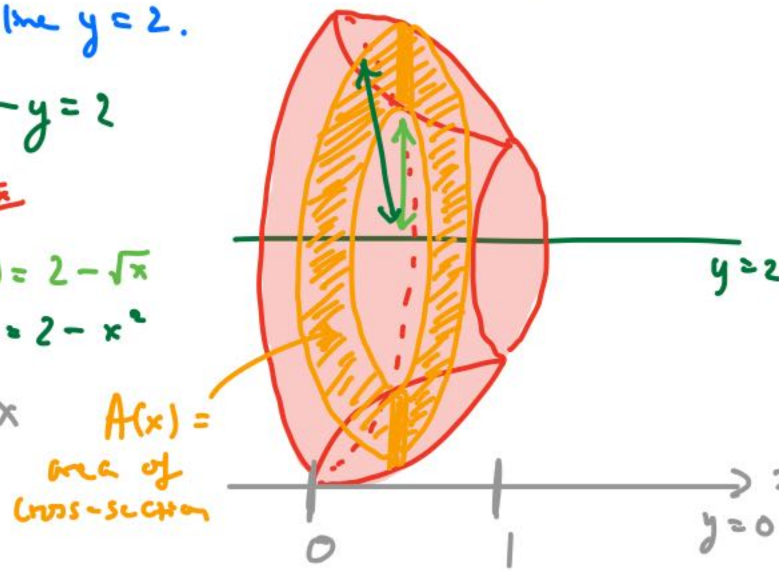


[Hint: use vertical disk slices, and $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$.]

Ans: $V = \int_0^\pi A(x)\,dx = \pi\int_0^\pi \sin(x)^2\,dx = \frac{\pi}{2}\int_0^\pi (1-\cos 2x)\,dx = \frac{1}{2}\pi^2$.

Ex 3/ Find the volume of the solid obtained by rotating the region bounded by $x=y^2$ & $y=x^2$ about the line $y=2$.





$$V = \int_0^1 A(x)\,dx$$

$$= \pi \int_0^1 \left(R(x)^2 - r(x)^2\right) dx = \pi \int_0^1 \left((2-x^2)^2 - (2-\sqrt{x})^2\right) dx$$

$$= \pi \int_0^1 \left[\cancel{4} - 4x^2 + x^4 - \cancel{4} + 4\sqrt{x} - x\right] dx$$

$$= \pi \left[-\frac{4}{3}x^3 + \frac{1}{5}x^5 + \frac{4x^{3/2}}{3/2} - \frac{x^2}{2}\right]_0^1$$

$$= \pi \left[-\frac{4}{3} + \frac{1}{5} + \frac{8}{3} - \frac{1}{2}\right) = \frac{31\pi}{30}.$$

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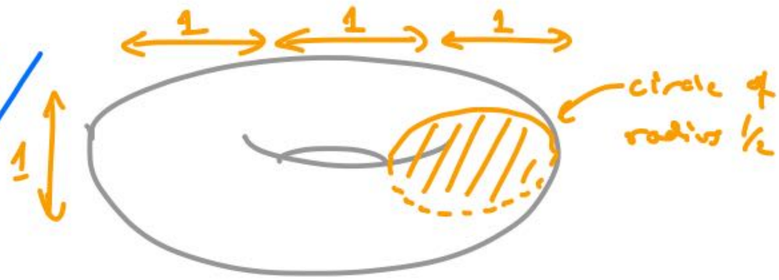
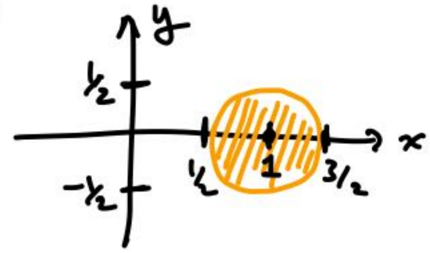
Since we haven't had enough desserts in this lecture, let's conclude with a yeast-raised donut.

Ex 4/



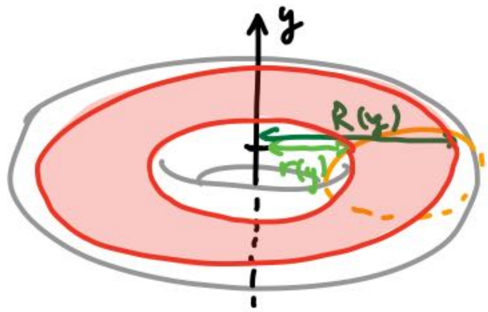
Notice that this is nothing but the solid of revolution obtained by rotating



about the y-axis. The equation of the circle is

$$(x-1)^2+y^2=(1/2)^2 \implies x-1=\pm\sqrt{1/4-y^2}$$
$$\implies x = 1 \pm \sqrt{1/4-y^2}.$$

So the inner & outer radii are



$$R(y) = 1+\sqrt{1/4-y^2}$$
$$r(y) = 1-\sqrt{1/4-y^2}$$

and

$$V = \pi\int_{-1/2}^{1/2}\left(R(y)^2 - r(y)^2\right)dy$$
$$= \pi\int_{-1/2}^{1/2}\left[\left(\cancel{1}+2\sqrt{1/4-y^2}+\cancel{\tfrac{1}{4}-y^2}\right)-\left(\cancel{1}-2\sqrt{1/4-y^2}+\cancel{\tfrac{1}{4}-y^2}\right)\right]dy$$
$$= 4\pi\int_{-1/2}^{1/2}\sqrt{1/4-y^2}\,dy$$

computed as area

Area = $\frac{1}{2}\pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{8}$

$$= 4\pi\cdot\frac{\pi}{8} = \frac{\pi^2}{2}.$$

(in other words, less filling than the square donut) //