Lecture 12: Volumes by shells

Let's begin with an example of volumes by <u>washers</u> to review the method we'll be offering an alternative to:

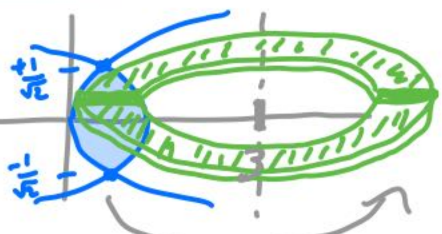
Try Write an integral which computes the volume of the solid of revolution obtained by revolving the region bounded by $x=y^2$ and $x=1-y^2$ about the line $x=3$.

Ans:

$y^2 = x = 1-y^2$

$2y^2 = 1$

$y = \pm\frac{1}{\sqrt{2}}$

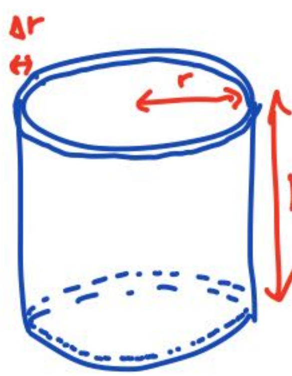


$$V = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} A(y)\,dy$$

$$= \pi \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \left(R(y)^2 - r(y)^2\right) dy$$

$$= \pi \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \left((3-y^2)^2 - (3-(1-y^2))^2\right) dy = \pi \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} (5-10y^2)\,dy$$

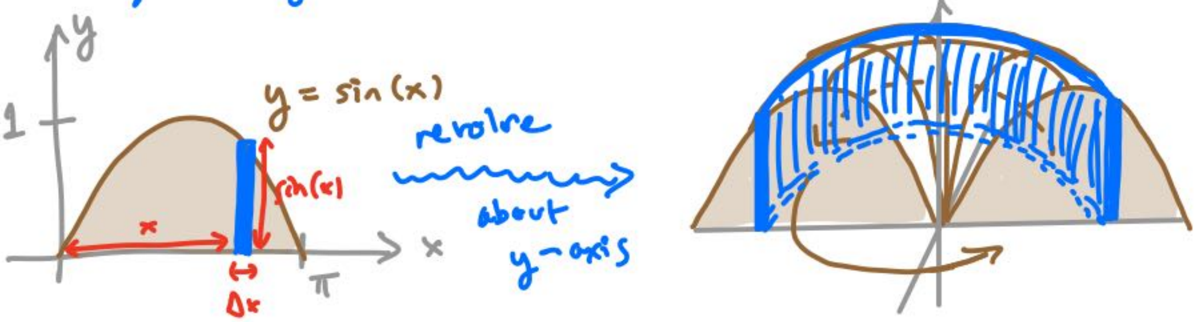
The shell method replaces thin washers by thin cylinders:

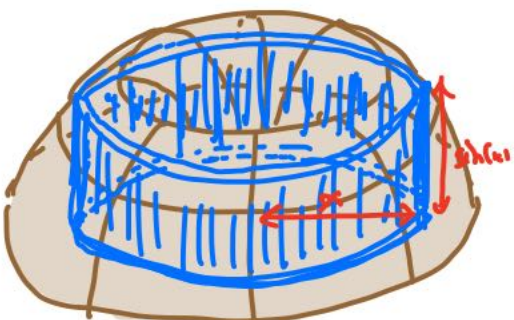


$\rightsquigarrow$ $\Delta V \approx 2\pi r h \cdot \Delta r$.

We'll compare it with the washer method on some of the same examples:

Ex 1 / (Trigonometric bundt cake)





$$\Delta V \approx 2\pi x \sin(x)\, \Delta x$$

By partitioning the interval $[0,\pi]$ we get a Riemann sum

$$V \approx \sum_{i=1}^{n} (\Delta V)_i \approx \sum_{i=1}^{n} 2\pi x_i \sin(x_i)\, \Delta x,$$

which yields (taking $n \to \infty$)

$$V = \int_0^{\pi} 2\pi x \sin(x)\, dx = 2\pi \Big[-x\cos(x) + \sin(x) \Big]_0^{\pi}$$

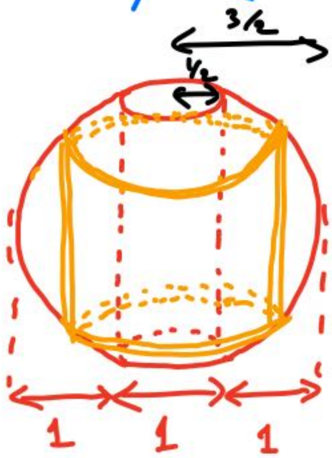
$$\left(\begin{aligned} &\frac{d}{dx}\big(-x\cos(x) + \sin(x)\big) \\ &= x\sin(x) - \cancel{\cos(x)} + \cancel{\cos(x)} \end{aligned} \right)$$

(this is a type of antidifferentiation we'll learn more about later)

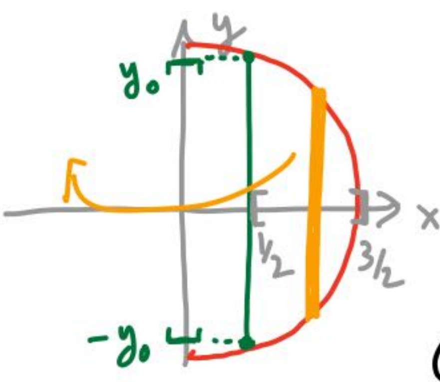
$$= 2\pi \Big[\big(-\pi \overbrace{\cos(\pi)}^{-1} + \underbrace{\cancel{\sin(\pi)}}_{0}\big) - \underbrace{\big(\cancel{-0\cos(0) + \sin(0)}\big)}_{0} \Big] = 2\pi^2.$$

Note that this looked much harder with washers. //

Ex 2 / (cored apple)



Here we get out our apple corers of various sizes and slice up the cored apple cylindrically!



We'll need the equation of the semi-circle again,

$$x^2 + y^2 = (3/2)^2,$$

and we allow the radius of the cylinder to run from 1/2 to 3/2. (We don't need to know $\pm y_0$!)

The height of the cylinder of radius x is then

$$h(x) = \sqrt{(3/2)^2 - x^2} - \left(-\sqrt{(3/2)^2 - x^2}\right) = 2\sqrt{9/4 - x^2}, \text{ and}$$

$$V = 2\pi \int_{1/2}^{3/2} x\, h(x)\, dx = 2\pi \int_{1/2}^{3/2} 2x \sqrt{9/4 - x^2}\, dx$$

$$= -2\pi \int_2^0 u^{1/2}\, du = 2\pi \int_0^2 u^{1/2}\, du = 2\pi \left[\frac{u^{3/2}}{3/2} \right]_0^2$$

$u = \frac{9}{4} - x^2$
$du = -2x\, dx$
$x = \frac{1}{2} \Rightarrow u = 2$; $x = \frac{3}{2} \Rightarrow u = 0$

$$= 2\pi \cdot \frac{2}{3} \cdot 2^{3/2} = \frac{8\sqrt{2}}{3}\pi .$$

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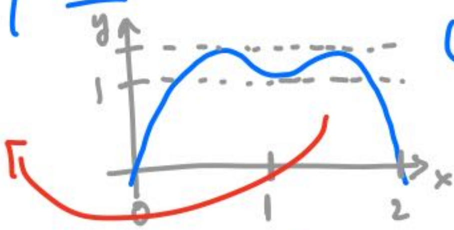
$\boxed{\text{Try}}$ Find the volume of the solid obtained by rotating the region bounded by $y = -2x^4 + 8x^3 - 11x^2 + 6x$ and the x-axis about the y-axis. [use a calculator!]

Ans: The intersection points are $(0,0)$ and $(2,0)$:



(this would be essentially impossible with washers, as you'd have to solve for x as a function of y!)

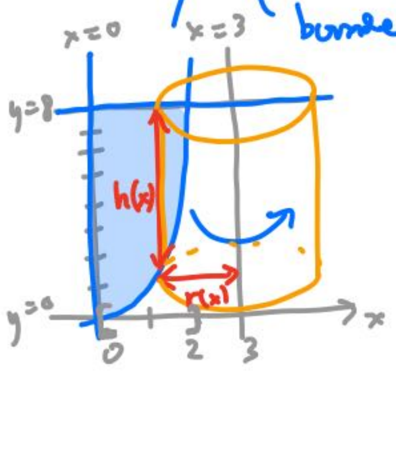
$$V = 2\pi \int_0^2 x \overbrace{\left(-2x^4 + 8x^3 - 11x^2 + 6x\right)}^{h(x)} dx = 2\pi \int_0^2 \left(-2x^5 + 8x^4 - 11x^3 + 6x^2\right) dx$$

$$= 2\pi \left[-\frac{x^6}{3} + \frac{8}{5}x^5 - \frac{11}{4}x^4 + 2x^3 \right]_0^2 = 2\pi \left[-\frac{64}{3} + \frac{256}{5} - 44 + 16 \right]$$

$$= \frac{56\pi}{15}.$$

As before, we can tackle problems involving rotation about some other axis:

Ex 3/ (Volume of solid generated by rotating the region bounded by $y = x^3$, $y = 8$, $x = 0$ about $x = 3$)



$$V = \int_0^2 2\pi r(x) h(x)\, dx = 2\pi \int_0^2 (3-x)(8-x^3)\, dx$$

$$= 2\pi \int_0^2 \left(24 - 8x - 3x^3 + x^4\right) dx = \frac{264\pi}{5}$$

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