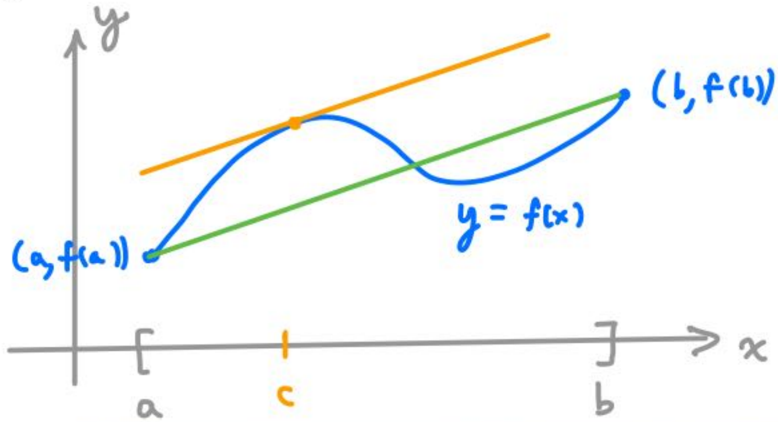
Lecture 13 : Some "average" work

Does everyone remember the MVT (= Mean Value Theorem)?

(Does *anyone* remember the MVT?)



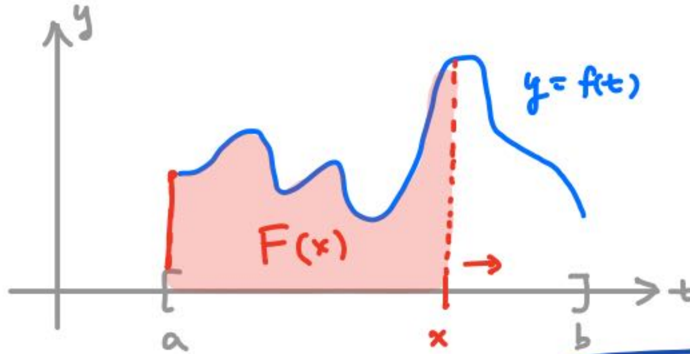
<u>MVT</u>: If f is differentiable on $[a,b]$, then there exists $c \in [a,b]$ at which

slope(secant line thru (a, f(a)) and (b, f(b))) = slope(tangent line)

i.e.
$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

So if your friend who drove 180 miles in 2 hours claims he was never going exactly 90 (at any instant), you know he's full of crap.

Let's try this out on $F(x) = \int_a^x f(t)\,dt$:



MVT $\Rightarrow$ $\exists$ $c \in [a,b]$ such that $\dfrac{F(b)-F(a)}{b-a} = F'(c)$. $(*)$

Now, $F'(c) = f(c)$, while
$F(b)-F(a) = \int_a^b f(t)\,dt$ $\Big\}$ 2 versions of FTC.

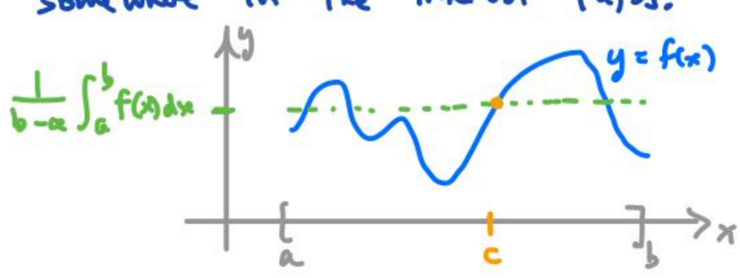
So $(*)$ just says

$$\boxed{\frac{1}{b-a}\int_a^b f(t)\,dt \underset{(**)}{=} f(c)} \quad \left(\begin{array}{c}\text{MVT for}\\ \text{integrals}\end{array}\right)$$

i.e. the height of f at some point c is equal to the left-hand side of $(**)$.

What is this LHS of $(**)$? It's the Mean Value (or average value) of f on $[a,b]$. So the MVT says:

Any function f achieves its average value on $[a,b]$, somewhere in the interval $[a,b]$:



To see that $\frac{1}{b-a}\int_a^b f(x)\,dx$ does indeed compute the average value of f on $[a,b]$, subdivide $[a,b]$ into n pieces with each $\Delta x = \frac{b-a}{n}$. The average of the n values

$$f(x_1),\ f(x_2),\ \ldots,\ f(x_n)$$

is

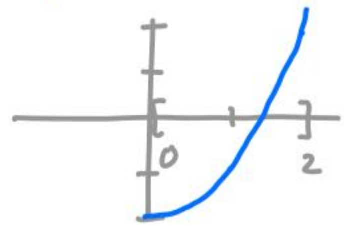
$$\frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n} = \sum_{i=1}^{n} f(x_i)\,\frac{1}{n}$$

$$= \frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\,\frac{b-a}{n}$$

$$= \frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\,\Delta x \qquad \text{Riemann sum}$$

$$\xrightarrow[\text{as } n\to\infty]{\text{take limit}} \frac{1}{b-a}\int_a^b f(x)\,dx\,.$$

Ex 1/ Find the average value of $f(x)=x^2-2$ over the interval $[0,2]$:



$$f_{avg} = \frac{1}{2-0}\int_0^2 (x^2-2)\,dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3} - 2x\right]_0^2$$

$$= \frac{1}{2}\left\{\left(\frac{2^3}{3} - 2\cdot 2\right) - (0-0)\right\}$$

$$= \frac{1}{2}\left(\frac{8}{3} - 4\right) = -\frac{2}{3}.$$

//

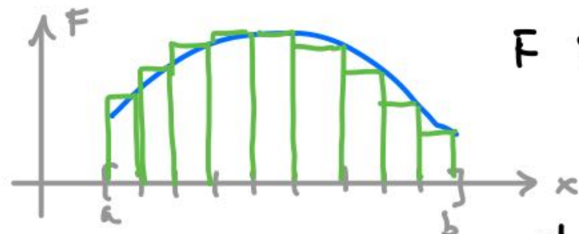
WORK If a constant force F is exerted on an object over a distance d, then the work done is

$$W = F \times d \qquad (\text{Newton} = kg \cdot m/s^2)$$

units: Joule = Newton × meter

(For example, if you lift an object weighing 1 Newton, 1 meter into the air, you are exerting an upward force of 1 Newton to counteract gravity, hence do 1 J of work.)

If the force varies over the distance interval $[a,b]$, we can subdivide $[a,b]$ into subintervals of width $\frac{b-a}{n} = \Delta x$ on which F is approximately constant.



We then have

$$W \approx \sum_{i=1}^{n} F(x_i)\,\Delta x \xrightarrow[n\to\infty]{} W = \int_a^b F(x)\,dx.$$

Ex 2/ A force of 3 N is required to hold a spring stretched 2 m beyond its natural length. How much work is done in stretching it from its natural length to 4 m beyond that?

Hooke's law: $F(x) = kx$ (k = "spring constant", x = distance beyond natural length)

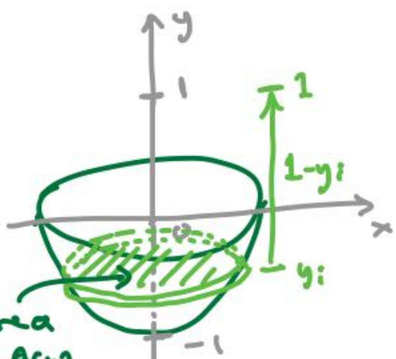
info. given $\Rightarrow 3\,N = k \cdot 2m \Rightarrow k = \frac{3}{2} N \cdot m^{-1}$.

So $W = \int_0^4 F(x)\,dx = \int_0^4 \frac{3}{2}x\,dx = \left[\frac{3}{4}x^2\right]_0^4 = 12\,J.$ //

So far this has been about moving "all of an object the same distance". What if the object is a rope being pulled, or a liquid being pumped, up to a specific height, so that some of the "object" is moving further than the rest?

Ex 3/

Bala Yaga's water tower is spherical (of radius 1 m) and half full of potion. Vasilisa has been asked to pump it all out of the opening in the top. How much work does her doll have to do if the potion has weight density 10000 N/m^3?



We "slice" the potion into disks.

Each disk must be raised from some height $y_i \in [-1, 0]$ to height $y = 1$.

So $$W \approx \sum_{i=1}^{n} \underbrace{(1-y_i)}_{\text{distance}} \underbrace{(\underbrace{A(y_i)\cdot \Delta y}_{\text{volume}} \cdot 10^4)}_{\text{weight (= upward force required to counteract gravity)}}$$

$\Big\downarrow n \to \infty$

$$W = \int_{-1}^{0} (1-y)\left(10^4 \pi (1-y^2)\right) dy$$

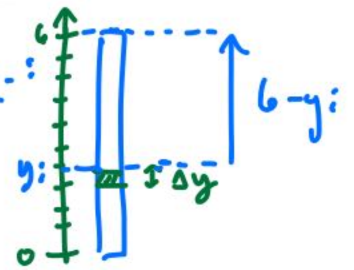
$$= 10^4 \pi \int_{-1}^{0} (1 - y - y^2 + y^3)\, dy = 10^4 \pi \left[y - \frac{y^2}{2} - \frac{y^3}{3} + \frac{y^4}{4} \right]_{-1}^{0}$$

$$= -10^4 \pi \left(-1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \right) = 10^4 \cdot \pi \cdot \frac{11}{12}.$$

//

Try You are dangling a 6-ft. colleague out of the window due to a professional disagreement. If each foot weighs 30 lbs., how much work is required to pull all 6 ft. of him up through the window? (Answer in ft-lbs. rather than Joules.)

Ans.:



30 lbs/ft. (linear density)

in ft.

$$W \approx \sum_{i=1}^{n} (30\,\Delta y)\cdot (6-y_i)$$

$\downarrow$ $n \to \infty$

$$W = 30\int_0^6 (6-y)\,dy = 30\left[6y - \frac{y^2}{2}\right]_0^6$$

$$= 30\,[36-18] = 30\cdot 18$$

$$= 540 \text{ ft-lbs.}$$