Lecture 14: Integration by Parts

Back to "how to compute integrals". Let $u(x)$, $v(x)$ be functions of x.

Product Rule: gives derivative of their product

$$u'(x)\,v(x) + u(x)\,v'(x) = \left(u(x)\,v(x)\right)'.$$

We successfully "reversed" the Chain Rule to get u-substitution for integrals. When we try to "reverse" the Product Rule, we get something called integration by parts:

$$\int \left(u'(x)v(x) + u(x)v'(x)\right) dx = u(x)\cdot v(x) + C$$

$$\Rightarrow \int u'(x)\,v(x)\,dx + \int u(x)\,v'(x)\,dx = u(x)\,v(x) + C$$

$$\Rightarrow \int u(x)\,v'(x)\,dx = u(x)\,v(x) - \int u'(x)\,v(x)\,dx,$$

absorbed into here

or more compactly

$$\int u v' \, dx = uv - \int v u' \, dx.$$

By using the shorthand $\begin{cases} du = \dfrac{du}{dx}\,dx = u'\,dx \\ dv = \dfrac{dv}{dx}\,dx = v'\,dx \end{cases}$

we obtain

$$\boxed{\int u\,dv = uv - \int v\,du} \quad \left(\int \text{by parts (I)}\right)$$

The trick in applying this method is in deciding which part of the original integral should be u and which dv.

Ex 1/ $$\int x \ln x \, dx = \frac{x^2}{2}\ln(x) - \int \frac{x^2}{2}\frac{dx}{x}$$

$$\left[d \left\{ \begin{array}{ll} u = \ln(x), & dv = x\,dx \\ du = \frac{dx}{x}, & v = x^2/2 \end{array} \right\} \int \right]$$

$$= \frac{x^2}{2}\ln(x) - \int \frac{x}{2}\,dx = \frac{x^2}{2}\ln(x) - \frac{x^2}{4} + C.$$

$$\left[\text{Check: } \frac{d}{dx}\left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right] = x\ln x + \cancel{\frac{x^2}{2}\frac{1}{x} - \frac{x}{2}} = x\ln x. \right] //$$

Ex 2/ $$\int \ln x \, dx = x \ln x - \int x \frac{dx}{x}$$

$$\left[\begin{array}{ll} u = \ln(x), & dv = dx \\ du = dx/x, & v = x \end{array} \right] \qquad = x\ln x - \int dx$$

$$= x\ln x - x + C. \qquad //$$

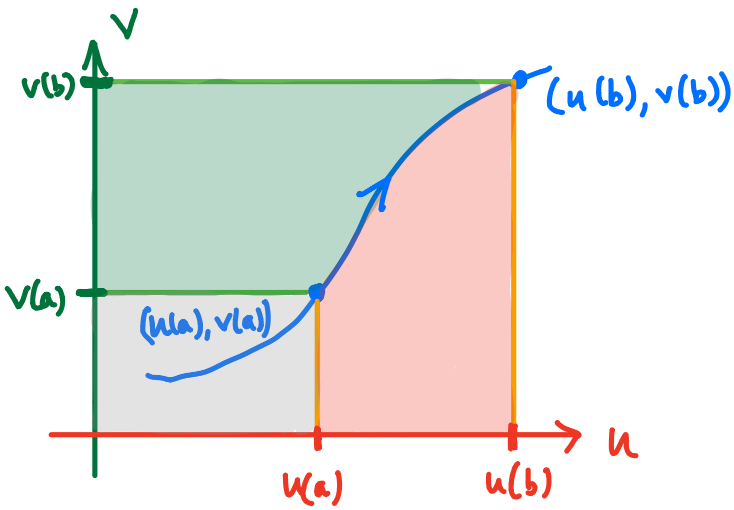
MAIN POINTS: ① The dv needs to be something that you already know how to integrate.

② $\int v\,du$ had better be easier than $\int u\,dv$!

Otherwise – choose different u, dv!

Now that we've dealt with indefinite integrals, we need to find a version of Integration by parts for <u>definite</u> integrals.

Let $x \longmapsto (u(x), v(x))$ parametrize a curve:



Think of x as time, so that u & v are the coordinates of an airplane, which starts at $(u(a), v(a))$ & ends at $(u(b), v(b))$. Having drawn the curve, we can now also think of it as expressing $\underbrace{u \text{ as a function of } v}_{u(v)}$, $\underbrace{\text{or vice versa}}_{v(u)}$.

Now consider the area of the big rectangle, which is the sum of areas of the three shaded regions:

$$\int_{u(a)}^{u(b)} v(u)\, du + \int_{v(a)}^{v(b)} u(v)\, dv + u(a)v(a) = u(b)v(b)$$

$$\Rightarrow \int_{v(a)}^{v(b)} u\, dv = uv \Big|_a^b - \int_{u(a)}^{u(b)} v\, du.$$

Now if you interpret $\begin{cases} dv \text{ as } v'\,dx \\ du \text{ as } u'\,dx \end{cases}$, so that the integral is actually over x (as is the case in practice), then x goes from a to b:

$$\boxed{\int_a^b u\, dv = uv \Big|_a^b - \int_a^b v\, du.} \quad \left(\int \text{ by parts (II)}\right)$$

(with the caveat that the integrals being performed are $\int_a^b uv'\,dx$, $\int_a^b vu'\,dx$).

Ex 3/ $\int_0^1 x e^{-x}\,dx \;=\; x(-e^{-x})\Big|_0^1 - \int_0^1 (-e^{-x})\,dx$

$$\begin{bmatrix} u = x, & dv = e^{-x}\,dx \\ du = dx, & v = -e^{-x} \end{bmatrix}$$

Why not $u = e^{-x}$ and $dv = x\,dx$? because $v = \frac{x^2}{2}$ then makes it worse!

$$= xe^{-x}\Big|_1^0 + \int_0^1 e^{-x}\,dx = 0 - 1e^{-1} + \left[-e^{-x}\right]_0^1$$

$$= -\frac{1}{e} + \left[e^{-x}\right]_1^0 = -\frac{1}{e} + e^0 - e^{-1} = 1 - \frac{2}{e}. \quad //$$

Ex 4/ $\int x^2 e^x\,dx \;=\; x^2e^x - \int e^x \cdot 2x\,dx$

$$\begin{bmatrix} u = x^2, & dv = e^x\,dx \\ du = 2x\,dx, & v = e^x \end{bmatrix}$$

$$= x^2e^x - \left(2xe^x - \int e^x \cdot 2\,dx\right)$$

$$\begin{bmatrix} u = 2x, & dv = e^x\,dx \\ du = 2\,dx, & v = e^x \end{bmatrix}$$

$$= x^2e^x - 2xe^x + 2e^x + C$$

$$= (x^2 - 2x + 2)e^x + C.$$

If you had been given $\int_0^1 x^2e^x\,dx$, you'd do the above (foreseeing this difficulty) and then write

$$\int_0^1 x^2e^x\,dx = \left[(x^2-2x+2)e^x\right]_0^1$$

$$= (1-2+2)e^1 - (0-0+2)e^0$$

$$= e - 2. \quad //$$

Repeated integration by parts is not uncommon. Here's another:

Ex 5/ Find $\int e^x \sin x \, dx$. Start with $\begin{bmatrix} u = e^x & dv = \sin(x)dx \\ du = e^x dx & v = -\cos(x) \end{bmatrix}$

$$\rightsquigarrow \int e^x \sin(x)\,dx = -e^x\cos(x) + \int e^x \cos(x)\,dx$$

$\begin{bmatrix} u = e^x & dv = \cos(x)dx \\ du = e^x dx & v = \sin(x) \end{bmatrix}$

$$= -e^x\cos(x) + e^x\sin(x) - \int e^x \sin(x)\,dx$$

!!

$$\Rightarrow 2\int e^x \sin(x)\,dx = -e^x\cos(x) + e^x\sin(x) + C$$

$$\Rightarrow \int e^x \sin(x)\,dx = \tfrac{1}{2}e^x(\sin(x) - \cos(x)) + K. \;//$$

Sometimes you need to substitute then "$\int$ by parts":

Ex 6/ $\int e^{\sqrt{x}}\,dx = \int e^t \cdot 2t\,dt$

$$\left(\begin{aligned} t &= \sqrt{x} = x^{1/2} \\ dt &= \tfrac{1}{2}x^{-1/2}dx = \frac{1}{2\sqrt{x}}dx = \frac{1}{2t}dx \\ \Rightarrow 2t\,dt &= dx \end{aligned} \right)$$

$$= 2te^t - \int e^t \cdot 2\,dt = 2te^t - 2e^t + C$$

$\begin{bmatrix} u = 2t & dv = e^t dt \\ du = 2dt & v = e^t \end{bmatrix}$

$$= 2(t-1)e^t + C = 2(\sqrt{x}-1)e^{\sqrt{x}} + C \;//$$

$\boxed{\text{Try}}$ $\int_1^e \ln x^2\,dx$, $\int \arcsin(x)\,dx$

Ans.: $e+1$, $x\arcsin(x) + \sqrt{1-x^2} + C$