Lecture 15: Trig + Inverse Trig Functions

Recall the main integration by parts formula

$$\int u\,dv = uv - \int v\,du$$

dv: $v'dx$ $\qquad$ du: $u'dx$

Try $\int_1^e x^2\ln(x)\,dx$

Ans: $= \left[\frac{1}{3}x^3\ln(x)\right]_1^e - \frac{1}{3}\int_1^e x^2\,dx = \frac{1}{3}\left(e^3 \cancel{\ln(e)}^{\,1} - 1\cancel{\ln(1)}^{\,0}\right) - \frac{1}{3}\left[\frac{x^3}{3}\right]_1^e$

$\left(\begin{array}{ll} u = \ln(x) & dv = x^2\,dx \\ du = \frac{1}{x}dx & v = \frac{x^3}{3} \end{array}\right)$

$= \frac{e^3}{3} - \frac{e^3}{9} + \frac{1}{9} = \frac{2e^3+1}{9}.$

While this doesn't always work, a reasonable rule of thumb is the following:

Strategy for $\int$ by Parts: Choose as u the function of the first type in the following list:

"LIATE"

differentiation greatly simplifies — easiest to integrate (as dv)

- L: log
- I: inverse trig
- A: algebraic (e.g. polynomial)
- T: trig
- E: exponential

(So for $\int x^2 \sin(x)\,dx$, you'd pick $u = x^2$, which results in $\int x \cos(x)\,dx$; you'd then pick $u = x$ and apply $\int$-by-parts once more.) But when the trig/exp functions have arguments other than $c \cdot x$, use substitution first:

Ex 1/ $$\int \frac{x^3 e^{x^2}}{(1+x^2)^2}\,dx \underset{\uparrow}{=} \int \frac{y e^y}{(1+y)^2} \cdot \frac{1}{2}\,dy.$$ $$\begin{pmatrix} y = x^2 \\ dy = 2x\,dx \end{pmatrix}$$

At this point, just applying the rule gets us into trouble. Instead take $u = ye^y$, $dv = \frac{dy}{(1+y)^2}$ $\rightsquigarrow$ $du = (y+1)e^y dy$, $v = \frac{-1}{1+y}$, so that the $\int$ becomes

$$-\frac{1}{2}\frac{ye^y}{1+y} + \frac{1}{2}\int \cancel{\frac{y+1}{1+y}}\, e^y\,dy = -\frac{1}{2}\frac{ye^y}{1+y} + \frac{1}{2}e^y + C = -\frac{1}{2}\frac{x^2 e^{x^2}}{1+x^2} + \frac{1}{2}e^{x^2} + C.$$ //

Ex 2/ $$\int \arcsin(x)\,dx \underset{\uparrow}{=} x \arcsin(x) - \int \frac{x\,dx}{\sqrt{1-x^2}}$$

$$\begin{pmatrix} t = 1-x^2 \\ dt = -2x\,dx \end{pmatrix} \qquad \begin{pmatrix} u = \arcsin(x) & dv = dx \\ du = \frac{dx}{\sqrt{1-x^2}} & v = x \end{pmatrix}$$

$$= x\arcsin(x) + \frac{1}{2}\int t^{-\frac{1}{2}}\,dt = x\arcsin(x) + \cancel{\frac{1}{2}}\frac{t^{1/2}}{\cancel{1/2}} + C = x\arcsin(x) + \sqrt{1-x^2} + C.$$ //

You may recall that we got tripped up computing volumes by this integral; $\int \arctan(x)\,dx = x\arctan(x) - \frac{1}{2}\ln(1+x^2) + C$ can be done in the same way.

Ex 3/ $$\int \sin^3(x)\,dx = -\sin^2(x)\cos(x) + 2\int \sin(x)\cos^2(x)\,dx$$

$$\begin{pmatrix} u = \sin^2(x) & , & dv = \sin(x)\,dx \\ du = 2\sin(x)\cos(x)\,dx & , & v = -\cos(x)\,dx \end{pmatrix}$$

$$= -\sin^2(x)\cos(x) - \underbrace{2\int t^2\,dt}_{\frac{2}{3}t^3} = -\sin^2 x\cos x - \frac{2}{3}\cos^3 x + C. \quad //$$

$$\begin{pmatrix} t = \cos(x) \\ dt = -\sin(x)\,dx \end{pmatrix}$$

Ex 4/ $$\int \cos^4(x)\,dx = \cos^3 x \sin x + 3\int \cos^2 x \sin^2 x\,dx$$

$$\begin{pmatrix} u = \cos^3 x & , & dv = \cos x\,dx \\ du = -3\cos^2 x \sin x\,dx & , & v = \sin x \end{pmatrix}$$

$$(\sin^2 x = 1 - \cos^2 x) \longrightarrow \; = \cos^3 x \sin x + 3\int \cos^2 x\,dx - 3\int \cos^4 x\,dx$$

$$\Longrightarrow 4\int \cos^4 x\,dx = \cos^3 x \sin x + 3\int \cos^2 x\,dx$$

$$\Longrightarrow \int \cos^4 x\,dx = \frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\int \cos^2 x\,dx$$

at which point we could repeat the above approach ($u = \cos x$, $v = \cos x\,dx$, etc.) or use a half-angle formula to get

$$\int \cos^2 x\,dx = \int\left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)dx = \frac{x}{2} + \frac{1}{4}\sin 2x + C.$$

Hence $$\int \cos^4 x\,dx = \frac{1}{4}\cos^3 x \sin x + \frac{3}{16}\sin 2x + \frac{3}{8}x + C. \quad //$$

In this way we can even obtain iterative formulas like

$$\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

from Stewart. But speaking of half-angle formulas, maybe there's a much easier way to do the last two examples?

Ex 3'/ $\int \sin^3 x \, dx \overset{(\sin^2 x = 1-\cos^2 x)}{=} \int (1-\cos^2 x)\sin x \, dx$

$= \int (t^2-1)\, dt = \frac{t^3}{3} - t + C$

$\begin{pmatrix} t = \cos x \\ -dt = \sin x \, dx \end{pmatrix}$ $\quad = \frac{1}{3}\cos^3 x - \cos x + C$. //

Ex 4'/ $\int \cos^4 x \, dx = \frac{1}{4}\int (1+\cos 2x)^2 \, dx$

$\left(\cos^2 2x = \frac{1}{2} + \frac{1}{2}\cos 4x\right)$ $\quad \left(\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x\right)$

$= \int \left(\frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x\right) dx$

$= \int \left(\frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{8} + \frac{1}{8}\cos 4x\right) dx$

$= \int \left(\frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x\right) dx$

$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$ ...

(which looks different from the answer to Ex. 4 !?

But it is in fact the same: $\frac{1}{32}\sin 4x = \frac{1}{16}\sin 2x \cos 2x$

$= \frac{1}{8}\sin x \cos x \,(2\cos^2 x - 1) = \frac{1}{4}\cos^3 x \sin x - \underbrace{\frac{1}{8}\sin x \cos x}_{\frac{1}{16}\sin 2x}$

$\Rightarrow \frac{1}{32}\sin 4x + \frac{1}{4}\sin 2x = \frac{1}{4}\cos^3 x \sin x + \frac{3}{16}\sin 2x$.) //

Notice that these methods completely avoid integration by parts.

Which leads us to the following

$$\boxed{\text{Strategy for } \int \sin^m x \cos^n x \, dx}$$

- If m, n even then use the half-angle formulas $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$, $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$ to reduce the degrees.
- If m or n is odd, say $= 2k+1$, then apply the Pythagorean theorem to the $2k^{\text{th}}$ power (e.g. $\sin^{2k} = (\sin^2)^k = (1-\cos^2)^k = \dots$), then use the substitution rule.

Ex 5/ $\int \sin^3 x \cos^2 x \, dx = \int (1-\cos^2 x) \sin x \cos^2 x \, dx$

$$= \int \sin x \left(\cos^2 x - \cos^4 x\right) dx \underset{\uparrow}{=} \int (t^4 - t^2) \, dt$$

$$\begin{pmatrix} t = \cos x \\ dt = -\sin x \, dx \end{pmatrix}$$

$$= \frac{t^5}{5} - \frac{t^3}{3} + C$$

$$= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C. \quad //$$

$\boxed{\text{Try}}$ $\int_0^{\pi} \sin^4 x \, dx$

Ans: $= \frac{1}{4}\int_0^{\pi} (1 - 2\cos 2x + \cos^2 2x) \, dx$

$$= \frac{1}{4}\int_0^{\pi} \left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right) dx = \frac{1}{4}\left[\frac{3}{2}x - \cancel{\sin 2x} + \cancel{\frac{1}{8}\sin 4x}\right]_0^{\pi} = \frac{3}{8}\pi.$$