Lecture 16 : More trig integrals

Last time we discussed a strategy for evaluating integrals of the form $\int \sin^m x \cos^n x \, dx$:

- if m, n even, use the half-angle formulas
- otherwise, break the odd power into an even power and a $\int \begin{cases}\sin x \\ \cos x\end{cases} dx$, then use $\cos^2 x + \sin^2 x = 1$ to rewrite the even power, and finally do a u-substitution.

$\boxed{\text{Try}}$ $\int_0^{\pi/2} \sin^4 x \cos^5 x \, dx$

Ans: write $\int_0^{\pi/2} \sin^4 x \,(\cos^2 x)^2 (\cos x \, dx) =$

$$\int_0^{\pi/2} \sin^4 x \,(1-\sin^2 x)^2 (\cos x \, dx) = \leftarrow \begin{array}{l} u = \sin x \\ du = \cos x \, dx \end{array}$$

$$\int_0^1 u^4 (1-u^2)^2 \, du = \int_0^1 (u^4 - 2u^6 + u^8) \, du =$$

$$\left[\frac{u^5}{5} - \frac{2}{7}u^7 + \frac{u^9}{9}\right]_0^1 = \frac{1}{5} - \frac{2}{7} + \frac{1}{9} = \frac{8}{315}.$$

Here are a couple not covered by the method above.

Ex 1/ $\int \frac{dx}{1-\sin x} = \int \frac{(1+\sin x)\,dx}{1-\sin^2 x} = \int \frac{\sin x + 1}{\cos^2 x} dx$

$$= \int \sec x \tan x \, dx + \int \sec^2 x \, dx$$

$$= \sec x + \tan x + C \quad \left(= \frac{1+\sin x}{\cos x} + C \right) //$$

Ex 2/ $\displaystyle\int \frac{1-\sin x}{\cos x}\,dx = \underbrace{\int \sec x \, dx} - \underbrace{\int \tan x \, dx}$

$$= \int \overbrace{\sec x \frac{\sec x + \tan x}{\sec x + \tan x}}^{\sec^2 x + \sec x \tan x} dx - \int \frac{\sin x}{\cos x}\,dx$$

$t = \cos x$
$-dt = \sin x \, dx$
$s = \sec x + \tan x$
$ds = (\sec x \tan x + \sec^2 x)\,dx$

$$= \int \frac{ds}{s} + \int \frac{dt}{t}$$

$$= \ln|s| + \ln|t| = \ln|\sec x + \tan x| + \ln|\cos x| + C$$

$$= \underbrace{\ln|\sec x + \tan x|} - \underbrace{\ln|\sec x|} + C$$

$$= \ln|1 + \sin x| + C .$$

& $\displaystyle\frac{\tan x}{\sec x} = \frac{\left(\frac{\sin x}{\cos x}\right)}{\left(\frac{1}{\cos x}\right)} = \sin x$

Check: $\displaystyle\frac{d}{dx}\ln|1+\sin x| = \frac{\cos x}{1+\sin x} = \frac{\cos x\,(1-\sin x)}{(1+\sin x)(1-\sin x)}$

$$= \frac{\cos x\,(1-\sin x)}{1-\sin^2 x} = \frac{\cos x\,(1-\sin x)}{\cos^2 x} = \frac{1-\sin x}{\cos x} . \checkmark //$$

An important class of integrals is

$$\int \sin(mx)\sin(nx)\,dx , \quad \int \sin(mx)\cos(nx)\,dx ,$$

$$\int \cos(mx)\cos(nx)\,dx , \quad \text{where } m, n \in \mathbb{Z} .$$

I'll focus only on the first type, and refer you to the table at the end of §5.2 for the others.

Since

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B,$$

$$\cos(A-B) - \cos(A+B) = 2 \sin A \sin B$$

$$\Rightarrow \quad \sin A \sin B = \tfrac{1}{2}\cos(A-B) - \tfrac{1}{2}\cos(A+B).$$

$$\Rightarrow \quad \int \sin(mx)\sin(nx)\,dx = \frac{1}{2}\int \cos(mx-nx)\,dx - \frac{1}{2}\int \cos(mx+nx)\,dx$$

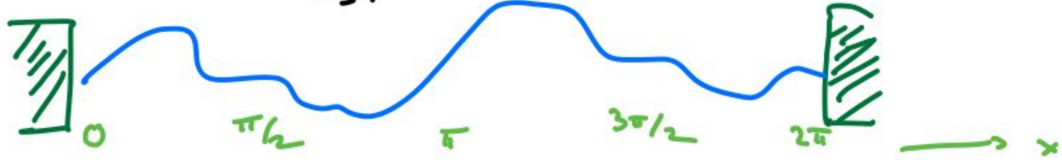
$$\begin{cases} \text{if } m \neq \pm n = \dfrac{1}{2(m-n)}\sin((m-n)x) - \dfrac{1}{2(m+n)}\sin((m+n)x) + C \\ \text{if } m = n \neq 0 = \dfrac{x}{2} - \dfrac{1}{4m}\sin(2mx) + C \\ \text{if } m = -n \neq 0 = \dfrac{1}{4m}\sin(2mx) - \dfrac{x}{2} + C \\ \text{if } m = n = 0 = C. \end{cases}$$

As for definite integrals, we therefore have

$$\int_0^{2\pi} \sin(mx)\sin(nx)\,dx = \begin{cases} 0 & \text{if } m \neq \pm n \text{ or } m = n = 0 \\ \pi & \text{if } m = n \neq 0 \\ -\pi & \text{if } m = -n \neq 0. \end{cases}$$

So suppose you had a string of length 2π, fixed at either end, and a wave on that string:

$$W(x) = \sum_{m \geq 1} a_m \sin(mx)$$



Suppose we don't know the amplitudes a_m of the various frequencies in the wave, and we'd like to know them. Then you ask a computer to evaluate

$$\frac{1}{\pi}\int_0^{2\pi} w(x)\sin(nx)\,dx = \frac{1}{\pi}\sum_{m\geq 1} a_m \int_0^{2\pi} \sin(mx)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\sum_{m\geq 1} a_m \begin{cases} 0 & m\neq n \\ \pi & m = n \end{cases}$$

$$= a_n \, .$$

Why do you care? Because often only the first few nonzero a_n are important for understanding the wave, allowing you to store much less information (data compression).

OK, here's one more class of integrals we're supposed to learn:

$$\int \tan^m x \sec^n x \, dx \, .$$

$\boxed{\text{Case 1}}$ ($n>0$ even) $\int \tan^m x \sec^{2k} x\, dx$

$$= \int \tan^m x (\sec^2 x)^{k-1} \sec^2 x \, dx = \int \tan^m x (1+\tan^2 x)^{k-1} \sec^2 x\, dx$$

$$= \int t^m (1+t^2)^{k-1}\, dt = \ldots$$

$t = \tan x$

$dt = \sec^2 x\, dx$

$\boxed{\text{Case 2}}$ (m odd, $n \geq 1$) $\int \tan^{2k+1} x \sec^n x \, dx$

$$= \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x \, dx$$

$$= \int (\sec^2 x - 1)^k \sec^{n-1} x \, (\sec x \tan x \, dx)$$

$$= \int (t^2 - 1)^k t^{n-1} \, dt = \cdots$$

$t = \sec x$

$dt = \sec x \tan x \, dx$

$\boxed{\text{Case 3}}$ ($n = 0$) $\int \tan^m x \, dx$

m odd: $\int \tan^{2k+1} x \, dx = \int (\sec^2 x - 1)^k \tan x \, dx$

$= \text{sum of Case 2 } \int\text{'s} \pm \underbrace{\int \tan x \, dx}_{\ln|\sec x|}$

m even: $\int \tan^{2k} x \, dx = \int (\sec^2 x - 1)^k \, dx$

$= \text{sum of Case 1 } \int\text{'s}.$

$\boxed{\text{Case 4}}$ (m even & n odd) $\int \tan^{2k} x \sec^{2l+1} x \, dx$

$= \int (\sec^2 x - 1)^k \sec^{2l+1} x \, dx$ = sum of $\int$'s of odd powers of sec.

i.e. reduces to $\underline{m = 0 \text{ and } n \text{ odd}}$: $\int \sec^{2l+1} x \, dx$

and apply $\int$-by-parts with $u = \sec x$, $\underbrace{dv = \sec^{2l} x \, dx}_{\text{requires doing this (Case 1)}}$.

Ex 3/ $\int \sec^3 x \, dx = \sec x \tan x - \int \underbrace{\tan^2 x}_{(\sec^2 x - 1)} \sec x \, dx$

$$\begin{pmatrix} u = \sec x & dv = \sec^2 x \, dx \\ du = \sec x \tan x \, dx & v = \tan x \end{pmatrix}$$

$$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$$\Rightarrow \quad 2 \int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx$$

$$= \sec x \tan x + \ln|\sec x + \tan x| + C$$

$$\Rightarrow \quad \int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C. \quad //$$