Lecture 17: Trig substitution

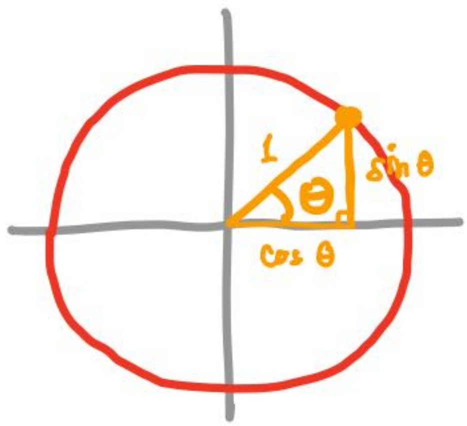
Begin by recalling the

Pythagorean Identities

$$\cos^2\theta + \sin^2\theta = 1 \qquad \frac{1}{\cos^2\theta}$$

$$\frac{1}{\sin^2\theta} \qquad 1 + \tan^2\theta = \sec^2\theta$$

$$\cot^2\theta + 1 = \csc^2\theta$$



These will be key to today's topic, as they were also key to evaluating integrals of the form $\int \tan^m\theta \sec^n\theta \, d\theta$ in the last one... speaking of which, let's briefly review:

Try $\displaystyle\int_0^{\pi/4} \sec^4 x \, dx$

Ans: $\displaystyle = \int_0^{\pi/4} (\tan^2 x + 1)\sec^2 x \, dx \underset{\substack{u = \tan x \\ du = \sec^2 x\, dx}}{=} \int_0^1 (u^2+1)\, du$

$$= \left[\frac{u^3}{3} + u\right]_0^1 = \frac{4}{3}.$$

Remember the general form of substitution:

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(t)\, dt .$$

$$\begin{pmatrix} t = g(x) \\ dt = g'(x)\, dx \end{pmatrix}$$

What if we try this "backwards"? i.e., suppose we have some integral $\int_A^B F(x)\, dx$ we can't do in its present form.

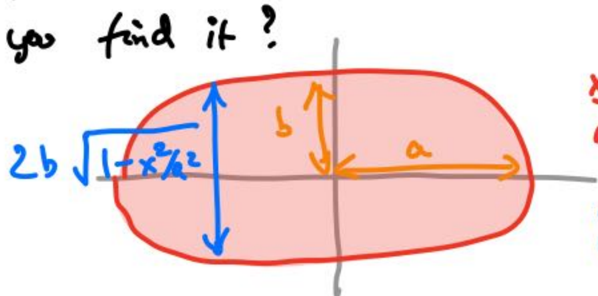
Now make the substitution

$$\int_A^B F(x)\, dx = \int_{G^{-1}(A)}^{G^{-1}(B)} F(G(t))\, G'(t)\, dt .$$

$$\begin{pmatrix} x = G(t) \\ dx = G'(t)\, dt \end{pmatrix}$$

Is there any reason to expect that this improves the situation?

Ex 1/ Suppose you didn't know the area of an ellipse, or for that matter, a circle. How would you find it?



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$y = \pm b \sqrt{1 - \frac{x^2}{a^2}}$$

We have to compute

$$(A =)\ 2b \int_{-a}^{a} \sqrt{1 - \frac{x^2}{a^2}}\, dx .$$

How do we do that? Recalling that $\sqrt{1-\sin^2\theta} = \cos\theta$, you might write

$$\int_{-a}^{a} \underbrace{\sqrt{1-\frac{x^2}{a^2}}}_{\text{Can't integrate as it stands!!}} \, dx = \int_{-\pi/2}^{\pi/2} \sqrt{1-\frac{\cancel{a^2}\sin^2\theta}{\cancel{a^2}}} \, a\cos\theta \, d\theta$$

$$\begin{pmatrix} x = a\sin(\theta) \\ dx = a\cos(\theta)\,d\theta \\ x = \pm a \iff \theta = \pm\frac{\pi}{2} \end{pmatrix}$$

$$= a\int_{-\pi/2}^{\pi/2} \cos^2\theta \, d\theta = \frac{a}{2}\int_{-\pi/2}^{\pi/2} (1+\cos 2\theta)\, d\theta$$

$$= \frac{a}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\pi/2}^{\pi/2} = \frac{a}{2}\underbrace{\left[\left(\frac{\pi}{2}+0\right) - \left(\frac{-\pi}{2}+0\right)\right]}_{\pi}$$

$$= \frac{a\pi}{2}$$

$$\Rightarrow A = 2b \cdot \frac{a\pi}{2} = \pi ab,$$

with πr^2 for a circle as a special case! //

Ex 2/ Suppose you did not know (or did not remember) the antiderivative of $\frac{1}{\sqrt{1-x^2}}$. You might try:

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{(\cos\theta)\,d\theta}{\sqrt{1-\sin^2\theta}} = \int \frac{\cos\theta}{\cos\theta}\,d\theta$$

nasty! make this disappear!

$$\begin{pmatrix} x = \sin\theta \\ dx = \cos\theta\,d\theta \end{pmatrix}$$

$$= \int d\theta = \theta + C$$

$$= \arcsin(x) + C \quad //$$

Ex 3/ What about $\frac{1}{4+x^2}$? Perhaps you forget where the 4 goes in the argument of arctan, or forget it's arctan. But you might remember that $1+\tan^2 = \sec^2$, and write

$$\int \frac{dx}{4+x^2} \underset{\uparrow}{=} \int \frac{2\sec^2\theta\, d\theta}{4+4\tan^2\theta} = \int \frac{2\cancel{\sec^2\theta}\, d\theta}{4\cancel{(1+\tan^2\theta)}}$$

$$\begin{pmatrix} x = 2\tan\theta \\ dx = 2\sec^2\theta\, d\theta \end{pmatrix} \qquad = \frac{1}{2}\int d\theta = \frac{1}{2}\theta + C$$

$$\downarrow$$

$$\begin{pmatrix} \frac{x}{2} = \tan\theta \\ \arctan\left(\frac{x}{2}\right) = \theta \end{pmatrix} \longrightarrow = \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C. \quad //$$

Your textbook suggests the following table for determining when to make which trigonometric substitution:

expression	inverse substitution	domain	relevant trig identity
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$1-\sin^2\theta = \cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$1+\tan^2\theta = \sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$	$\sec^2\theta - 1 = \tan^2\theta$

why ?? - Because this is where $x \leftrightarrow \theta$ is 1-to-1

In general, when you make such substitutions, you may well end up with an integral of the type we learned to evaluate in §7.2.

$\boxed{\text{Try}}$ $\displaystyle\int_1^2 \frac{dx}{x^3\sqrt{x^2-1}}$

(Fact: Can you think of any other way to do it? Usual substitution / ∫-by-parts both useless!)

Ans: $\displaystyle = \int_0^{\pi/3} \frac{\sec\theta \cancel{\tan\theta}\, d\theta}{\sec^3\theta \cancel{\sqrt{\sec^2\theta - 1}}} = \int_0^{\pi/3} \frac{d\theta}{\sec^2\theta}$

$\left[\begin{array}{l} x = \sec\theta \left(= \frac{1}{\cos\theta}\right) \\ dx = \sec\theta\tan\theta\, d\theta \end{array} : \begin{array}{l} x = 2 \leftrightarrow \theta = \pi/3 \\ x = 1 \leftrightarrow \theta = 0 \end{array}\right]$ $\displaystyle = \int_0^{\pi/3} \cos^2\theta\, d\theta$

$\displaystyle = \frac{1}{2}\int_0^{\pi/3} (1+\cos 2\theta)\, d\theta = \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/3}$

$\displaystyle = \frac{1}{2}\left[\left(\frac{\pi}{3} + \frac{1}{2}\underbrace{\sin\left(\frac{2\pi}{3}\right)}_{\sqrt{3}/2}\right) - \underbrace{\left(0 + \frac{1}{2}\sin 0\right)}_{0}\right]$

$\displaystyle = \frac{\pi}{6} + \frac{\sqrt{3}}{8}.$

We'll look at a couple more types of these problems on Monday before turning to partial fractions.