Lecture 18 : Trig. Substitution II; Partial Fractions I

Continuing our discussion of trigonometric substitution, we push a bit beyond $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$, by the technique of COMPLETING THE SQUARE.

Ex 1/ Compute $\int \sqrt{5+4x-x^2}\,dx$.

Write
$$\begin{aligned} 5+4x-x^2 &= 5-(x^2-4x) \\ &= 5-(\underbrace{x^2-4x+4}-4) \\ &= 9-(x-2)^2. \end{aligned}$$

So
$$\begin{aligned} \int \sqrt{5+4x-x^2}\,dx &= \int \sqrt{9-(x-2)^2}\,dx \\ \left(\begin{aligned} u &= x-2 \\ du &= dx \end{aligned}\right) \rightarrow &= \int \sqrt{9-u^2}\,du \\ \left(\begin{aligned} u &= 3\sin\theta \\ du &= 3\cos\theta\,d\theta \end{aligned}\right) \rightarrow &= \int \sqrt{9-9\sin^2\theta}\;3\cos\theta\,d\theta \\ &= 9\int \sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta \\ &= 9\int \cos^2\theta\,d\theta \\ &= \frac{9}{2}\int (1+\cos 2\theta)\,d\theta \\ &= \frac{9}{2}\left(\theta+\frac{1}{2}\sin 2\theta\right)+C \\ &= \frac{9}{2}\theta+\frac{9}{4}\sin 2\theta + C \end{aligned}$$

Now we have to revert to x instead of θ or u:

$$= \frac{9}{2}\theta + \frac{9}{2}\sin\theta\cos\theta + C$$

$$\left(\begin{array}{l} x-2\,(=u) = 3\sin\theta \\ \frac{x-2}{3} = \sin\theta \end{array} \right)$$

$$= \frac{9}{2}\arcsin\left(\frac{x-2}{3}\right) + \frac{\overset{3}{\cancel{9}}}{2}\left(\frac{x-2}{\cancel{3}}\right)\cos\left(\arcsin\left(\frac{x-2}{3}\right)\right) + C$$

$$\theta = \arcsin A \Rightarrow \cos\theta = \sqrt{1-A^2}$$

$$= \frac{9}{2}\arcsin\left(\frac{x-2}{3}\right) + \frac{3(x-2)}{2}\sqrt{1-\left(\frac{x-2}{3}\right)^2} + C$$

$$= \frac{9}{2}\arcsin\left(\frac{x-2}{3}\right) + \frac{x-2}{2}\sqrt{5+4x-x^2} + C$$

//

Ex 2/ $\displaystyle\int \frac{dt}{\sqrt{t^2-6t+13}} = \int \frac{dt}{\sqrt{(t-3)^2+4}}$

$$\left(\begin{array}{l} t^2-6t+13 = t^2-6t+9+4 \\ \qquad = (t-3)^2+4 \end{array} \right)$$

(Let's skip the intermediate u-substitution:
$t-3 = 2\tan\theta$
$dt = 2\sec^2\theta\, d\theta$)

$$= \int \frac{\cancel{2}\sec^2\theta\, d\theta}{\sqrt{\cancel{4}\tan^2\theta + \cancel{4}^{\,1}}}$$

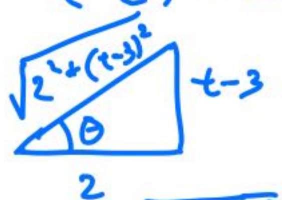
$$= \int \frac{\sec^2\theta\, d\theta}{\sqrt{\tan^2\theta+1}}$$

$$= \int \frac{\sec^{\cancel{2}}\theta\, d\theta}{\cancel{\sec\theta}} = \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$\frac{t-3}{2} = \tan\theta$$

$$\arctan\left(\frac{t-3}{2}\right) = \theta$$



$$\sec\theta = \frac{\sqrt{2^2+(t-3)^2}}{2} = \frac{\sqrt{t^2-6t+13}}{2}$$

$$= \ln\left|\frac{1}{2}\sqrt{t^2-6t+13} + \frac{t-3}{2}\right| + C.$$

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These were some tricky problems!

Introduction to Partial Fractions

Ex 3/ How would you integrate

$$\int \frac{3x-1}{x^2+x-2}\,dx \ ?$$

If the numerator was the derivative of the denominator, $2x+1$, or a multiple of it, we'd have

$$a\int \frac{g'(x)}{g(x)}\,dx = a\ln|g(x)| + C.$$

Unfortunately, this is not the case. But if you happen to notice that

$$\frac{3x-1}{x^2+x-2} = \frac{2/3}{x-1} + \frac{7/3}{x+2},$$

you're all set:

$$\int \frac{2/3}{x-1}\,dx + \int \frac{7/3}{x+2}\,dx = \frac{2}{3}\ln|x-1| + \frac{7}{3}\ln|x+2| + C.$$

But how did we know we could rewrite the rational function $\frac{3x-1}{x^2+x-2}$ in this way? This is what partial fractions is all about. First, you factor the denominator:

$$x^2+x-2 = (x-1)(x+2).$$

Then you write

unknown constants

$$(*) \qquad \frac{3x-1}{x^2+x-2} = \frac{A}{x-1} + \frac{B}{x+2}.$$

A & B have yet to be determined, so that $(*)$ holds.

(But notice that x^2+x-2 is the common denominator for the right-hand side.) Multiply both sides by $(x-1)(x+2)$:

$$3x-1 = A(x+2) + B(x-1)$$

$$(**) \qquad 3x + (-1) = (A+B)x + (2A-B).$$

For $(**)$ to hold for all x, we must have

$$\text{(i)}\ 3 = A+B \quad \text{and} \quad \text{(ii)}\ -1 = 2A-B.$$

[Check: $x = 0 \Rightarrow$ (ii), then cancel the constants from $(**)$ to get (i). More generally, if

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$$

for all x, then $a_n = b_n$, $a_{n-1} = b_{n-1}$, $\ldots$, $a_1 = b_1$, & $a_0 = b_0$.]

So we get a system of 2 equations:

(i) $\Rightarrow$ $B = 3 - A$ $\rightsquigarrow$ plug into (ii)

$$-1 = 2A - (3-A)$$

$$\Rightarrow 2 = 3A$$

$$\Rightarrow A = 2/3 \Rightarrow B = 3 - 2/3 = 7/3.$$

Ex 4/ Integrate $\displaystyle\int \frac{x^3 - x^2 - 7x + 2}{x^2 - 3x + 2}\,dx$.

Whoa! The top has higher degree than the bottom — it's an IMPROPER fraction. The thing is, the partial fractions technique from Example 3 only works for cases where deg (numerator) < deg (denominator).

So begin by long division:

$$
\begin{array}{r l}
 & x + 2 - \dfrac{3x+2}{x^2-3x+2} \\
x^2 - 3x + 2 \,\big) & x^3 - x^2 - 7x + 2 \\
 & -(x^3 - 3x^2 + 2x) \\ \hline
 & 2x^2 - 9x + 2 \\
 & -(2x^2 - 6x + 4) \\ \hline
 & -3x - 2 \quad \text{remainder}
\end{array}
$$

do partial fractions on this!

Now $x^2 - 3x + 2 = (x-2)(x-1)$. So write

$$\frac{3x+2}{x^2-3x+2} = \frac{A}{x-2} + \frac{B}{x-1}$$

$$
\begin{aligned}
3x + 2 &= A(x-1) + B(x-2) \\
&= (A+B)x + [-A - 2B]
\end{aligned}
$$

$$\Rightarrow \quad 3 = A + B, \quad 2 = -A - 2B$$

$\Rightarrow \quad 3 - B = A, \qquad 2 = -(3-B) - 2B$

$\qquad\qquad\qquad\qquad\qquad\quad = -3 - B$

$\Rightarrow \quad B = -5, \quad A = 3-(-5) = 8.$

[Check: $\dfrac{8}{x-2} - \dfrac{5}{x-1} = \dfrac{8(x-1) - 5(x-2)}{(x-2)(x-1)} = \dfrac{3x+2}{(x-2)(x-1)}$. ✓]

So the integral

$$\int \frac{x^3 - x^2 - 7x + 2}{x^2 - 3x + 2}\,dx = \int (x+2)\,dx - \int \frac{3x+2}{x^2-3x+2}\,dx$$

$$= \frac{x^2}{2} + 2x - \int \frac{8}{x-2}\,dx + \int \frac{5}{x-1}\,dx$$

$$= \frac{x^2}{2} + 2x - 8\ln|x-2| + 5\ln|x-1| + C. \quad //$$

$\boxed{\text{Try}}$ Find the quotient and remainder of $\dfrac{x^3 - 4x - 10}{x^2 - x - 6}$.

[Ans: quotient $= x+1$, remainder $= 3x-4$]

$\boxed{\text{Try}}$ Find the partial fraction decomposition of $\dfrac{3x-4}{x^2-x-6}$.

[Ans: $\dfrac{1}{x-3} + \dfrac{2}{x+2}$]