Lecture 19 : Partial Fractions II

On Monday we introduced a method for integrating rational functions $\frac{P(x)}{Q(x)}$ (P, Q polynomials) where $Q(x)$ takes the form $(x-a)(x-b)$ with $a \neq b$. What about a more "general" $Q(x)$?

Ex 1/ $\int \frac{dx}{x^2(x+1)}$ ... ? Since the x is <u>repeated</u>, we need to write

$$(*) \qquad \frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} .$$

[More generally, we'll encounter examples like

$$\frac{\text{blah}}{(x+2)^3(x-3)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3} + \frac{D}{(x-3)} + \frac{E}{(x-3)^2} .$$

This is the general pattern when the denominator can be factored into linear factors.]

Now $(*)$ becomes

$$Ax(x+1) + B(x+1) + Cx^2 = 1$$

$$Ax^2 + Ax + Bx + B + Cx^2 = 1$$

$$(A+C)x^2 + (A+B)x + B = 1 \quad (\text{for all } x!)$$

$$\Rightarrow \begin{cases} B = 1 \\ A+B = 0 \\ A+C = 0 \end{cases} \Longrightarrow \begin{cases} A = -1 \\ B = 1 \\ C = 1 \end{cases}.$$

So $$\int \frac{dx}{x^2(x+1)} = \int \left\{ -\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1} \right\} dx$$

$$= -\ln|x| - \frac{1}{x} + \ln|x+1| + C$$

$$= \ln\left|1+\frac{1}{x}\right| - \frac{1}{x} + C. \quad //$$

The key that makes the strategy we've been using so far effective is for the denominator to factor into linear factors. In this case, the integral becomes a sum of integrals of the form

$$\int \frac{\text{Const}}{(x-r)^k}\,dx = \begin{cases} \dfrac{\text{const}}{(1-k)(x-r)^{k-1}} & k > 1 \\ \text{Const} \cdot \ln|x-r| & k = 1. \end{cases}$$

Ex 2/ What about something like

$$\frac{6x^2-3x+1}{4x^3+x^2+4x+1} = \cdots \frac{6x^2-3x+1}{(4x+1)(x^2+1)}$$ doesn't break into linear factors

??? We can try to solve for constants A and B such that

$$\frac{6x^2-3x+1}{(4x+1)(x^2+1)} = \frac{A}{4x+1} + \frac{B}{x^2+1}$$

but this doesn't work: the right-hand side is

$$\frac{A(x^2+1)+B(4x+1)}{(4x+1)(x^2+1)} = \frac{Ax^2+4Bx+(A+B)}{(4x+1)(x^2+1)},$$

and $\begin{cases} 6 = A \\ -3 = 4B \\ 1 = A+B \end{cases}$ is insolvable. We need another constant C to play with: try $\frac{6x^2-3x+1}{(4x+1)(x^2+1)} =$

$$\frac{A}{4x+1} + \frac{Bx+C}{x^2+1} = \frac{(A+4B)x^2+(B+4C)x+(A+C)}{(4x+1)(x^2+1)}$$

$$\Rightarrow \begin{cases} 6 = A+4B \\ -3 = B+4C \\ 1 = A+C \end{cases} \xRightarrow[\text{eliminate } A]{} \begin{cases} 5 = 4B-C \\ -3 = B+4C \\ 12 = -4B-16C \end{cases} \times(-4)$$

$$\underset{\text{eliminate } B}{\Longrightarrow} 17 = -17C \Rightarrow C = -1 \Rightarrow B = 1$$

$$\Rightarrow A = 2. \quad \text{So}$$

$$\int \frac{6x^2-3x+1}{4x^3+x^2+4x+1}\,dx = \int \frac{2}{4x+1}\,dx + \int \frac{x-1}{x^2+1}\,dx$$

$$= \underbrace{\int \frac{\frac{1}{2}\,du}{u}}_{\ln|u|} + \int \frac{x}{x^2+1}\,dx - \int \frac{dx}{x^2+1}$$

$u = 4x+1$
$du = 4dx$

use
$t = x^2+1$
$dt = 2x\,dx$

$$= \frac{1}{2}\ln|4x+1| + \frac{1}{2}\ln(x^2+1) - \arctan(x) + C. \quad //$$

It may not always be so nice to integrate

$$\int \frac{Ax+B}{ax^2+bx+c}\,dx$$

when ax^2+bx+c is an irreducible quadratic. Consider

Ex 3/ $\displaystyle\int \frac{dx}{x^2-4x+13} \rightsquigarrow x_\pm = \frac{4 \pm \sqrt{16-52}}{2}$ negative! So polynomial is irreducible!

complete the square:
$x^2-4x+13 = x^2-4x+4+9 = (x-2)^2+9$

$$\int \frac{dx}{(x-2)^2+9} = \int \frac{3\,du}{9u^2+9} = \frac{1}{3}\arctan(u) + C$$

$3u = x-2$
$3\,du = dx$

$$= \frac{1}{3}\arctan\left(\frac{x-2}{3}\right) + C. \quad //$$

The "worst" case scenario is something like

Ex 4/ $\displaystyle\int \frac{2x+2}{x^2-4x+8}\,dx$ irreducible

First, we want to reduce this to $\dfrac{A}{x^2-4x+8}$ like in Example 3. Now, $\frac{d}{dx}(x^2-4x+8) = 2x-4$. So we can write

$$\int \frac{2x+2}{x^2-4x+8}\,dx = \int \frac{2x-4}{x^2-4x+8}\,dx + \int \frac{6}{x^2-4x+8}\,dx$$

$$= \int \frac{du}{u} + \int \frac{6}{(x-2)^2+4}\,dx$$

$u = x^2-4x+8$

$du = (2x-4)\,dx$

use $2t = x-2$

$2dt = dx$

$$= \ln|u| + \int \frac{\overset{3}{\cancel{12}}\,dt}{\cancel{4}t^2 + \cancel{4}_{1}}$$

$$= \ln|u| + 3\arctan(t) + C$$

$$= \ln|x^2-4x+8| + 3\arctan\left(\frac{x-2}{2}\right) + C. \quad //$$

The most general scenario in partial fractions decomposition is to have something like

$$(**) \qquad \frac{P(x)}{(x+3)(x-2)^2(x^2-4x+8)(x^2-4x+13)^2}.$$

polynomial

The degree of the denominator here is 9. If the degree $\deg(P(x)) \geq 9$, start by doing long division to replace it by a polynomial of degree < 9. Assuming $\deg(P(x)) < 9$, there exist constants A, B_i, C, D, E_i, F_i such that

$$(**) = \frac{A}{x+3} + \frac{B_1}{x-2} + \frac{B_2}{(x-2)^2} + \frac{Cx+D}{x^2-4x+8} + \frac{E_1x+F_1}{x^2-4x+13} + \frac{E_2x+F_2}{(x^2-4x+13)^2}.$$

$\boxed{\text{SUMMARY}}$ To decompose $f(x) = \frac{P(x)}{Q(x)}$

(1) If f improper ($\deg P \geq \deg Q$) do long division first

(2) Factor $Q(x)$

(3) Decompose fractions as above with unknown constants in numerators (over $(x-a)^k$, A ; over $(x^2-ax+b)^k$, $Ax+B$)

(4) Multiply thru by $Q(x)$ and get several equations by equating coefficients of like powers of x; solve for constants.

$\boxed{\text{Try}}$ Decompose $\frac{1}{(x+1)^2(x^2+1)}$ into partial fractions.

What are the constants A_1, A_2, B, C ?

Ans:
$$\frac{1}{(x+1)^2(x^2+1)} = \frac{A_1}{x+1} + \frac{A_2}{(x+1)^2} + \frac{Bx+C}{x^2+1}$$

$$\Rightarrow \quad 1 = A_1(x+1)(x^2+1) + A_2(x^2+1) + (Bx+C)(x+1)^2$$
$$= (A_1+B)x^3 + (A_1+A_2+2B+C)x^2 + (A_1+B+2C)x + (A_1+A_2+C)$$

$$\Rightarrow \begin{cases} A_1 + B = 0 \\ A_1 + A_2 + 2B + C = 0 \\ A_1 + B + 2C = 0 \\ A_1 + A_2 + C = 1 \end{cases} \Rightarrow \quad C = 0 \text{ and } \begin{cases} A_1 + A_2 + 2B = 0 \\ A_1 + B = 0 \\ A_1 + A_2 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} A_2 + B = 0 \\ 2B = -1 \end{cases} \Rightarrow B = -1/2 \Rightarrow A_2 = \tfrac{1}{2} \Rightarrow A_1 = \tfrac{1}{2}$$

$$\Rightarrow \text{get } \frac{1/2}{x+1} + \frac{1/2}{(x+1)^2} + \frac{-1/2\, x}{x^2+1}.$$