Lecture 2: Area under a curve

When Gauss was 7, his teacher thought he had a good problem to keep the class busy. He said, find
\[ 1 + 2 + 3 + \ldots + 100 = ? \]
In a few seconds Gauss wrote 5,050. How did he do it?
\[ 1 + 2 + 3 + \ldots + 50 + 51 + \ldots + 98 + 99 + 100 \]

50 pairs
each adds to 101
\[ 50 \times 101 = 5,050. \]

We can rewrite this sum more compactly in "sigma notation":
\[ \sum_{i=1}^{100} i = 5050. \]

Can we also find \( S_n = \sum_{i=1}^{n} i = 1 + \ldots + n \) for any \( n \)?

First we need some rules about sums:

\[ \sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n \]

\[ \sum_{k=1}^{n} k \times \text{sum of \( k \times \) the same } \]

as long as there are the same, makes no difference whether it's \( i, j, k, l \) etc. - called "dummy index".

**SUM RULES**

- \( \sum_{k=1}^{n} 1 = 1 + \ldots + 1 = n \)
- \( \sum_{k=1}^{n} (c \cdot a_k) = c \cdot \sum_{k=1}^{n} a_k \)
- \( \sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k \)
Now then: \((i+1)^2 - i^2 = 2i + 1\), right?

So \(\sum_{i=1}^{n} [(i+1)^2 - i^2] = \sum_{i=1}^{n} (2i + 1)\)

\[\sum_{i=1}^{n} (i+1)^2 - \sum_{i=1}^{n} i^2 = 2 \sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\]

\[-[i^2 + 2i^2 + \ldots + n^2 + (n+1)^2] \quad \text{collapsing sum!}\]

\[(n+1)^2 - 1^2 = 2S_n + n\]

\[n^2 + 2n + 1 - 1 = 2S_n + n\]

\[n^2 + 2n = 2S_n + n\]

\[\frac{n(n+1)}{2} = \frac{n^2 + n}{2} = S_n = \sum_{i=1}^{n} i = (1 \ldots + n)\]

You can do exactly the same thing, but with \((i+1)^2 - i^2 = 3i^2 + 3i + 1\)

at the beginning, to find that

\[\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}\]

If you want the details, try it yourself and/or look in Appendix E of your book.
Areas from limits of sums

Let's use this formula $(R)$ to find the area under a parabola, from 0 to 2:

\[ y = f(x) = x^2 \]

We can't compute the area of the parabola, so we'll approximate it with a polygon, and compute that area!

Begin by partitioning the interval $[0,2]$ into $n$ subintervals of length $\Delta x = \frac{2}{n}$.

So,

\[
\begin{align*}
  x_0 &= 0 \\
  x_1 &= \frac{2}{n} \\
  x_2 &= \frac{4}{n} \\
  \vdots \\
  x_i &= \frac{2i}{n} \\
  x_n &= \frac{2n}{n} = 2
\end{align*}
\]

Consider now the rectangle with base $[x_{i-1}, x_i]$ and height $f(x_i) = x_i^2$.

Its area is just

\[
A(R_i) = (\Delta x) \cdot x_i^2
\]

\[
= (\Delta x) \cdot f(x_i)
\]

\[
= \frac{2}{n} \cdot \left(\frac{2i}{n}\right)^2
\]

\[
= \frac{8}{n^3} \cdot i^2
\]
If we add the areas of these boxes up, we get a pretty good estimate of the area under the curve. This will be a slight overestimate, because we took the height of each box to be the largest value assumed by \( f(x) \) on the \( i \)th subinterval \([x_{i-1}, x_i]\).

**TERMINOLOGY:** this is called an upper sum, and yields an upper bound on the area. You could also use lower sums (to get a lower bound), or always take the height to be \( f(x_i) \) (right endpoint rule), \( f(x_{i-1}) \) (left endpoint rule), or \( f(\frac{x_i + x_{i-1}}{2}) \) (midpoint rule), or just any point in \([x_{i-1}, x_i]\).

Here the right endpoint rule & upper sum were the same, because \( f(x) \) is increasing, but that won’t always be the case!

Let’s do it! I’m going to add an extra index and call \( R_i \) instead of \( R^n \)

and \( R^n \) will be the union of all these boxes as \( i \) runs from 1 to \( n \). So

\[
A(R^n) = \sum_{i=1}^{n} A(R_i) = \frac{8}{n^3} \sum_{i=1}^{n} i^2
\]

\[
= \frac{8}{n^3} \sum_{i=1}^{n} i^2 = \frac{8}{n^3} \frac{n(n+1)(2n+1)}{6}
\]
\[
\frac{4}{3} \cdot \frac{2n^3 + 3n^2 + n}{n^3} = 4 \left( 2 + \frac{3}{n} + \frac{1}{n^2} \right) = \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}.
\]

As \( n (= \# \text{ of boxes}) \to \infty \), \( \Delta x \to 0 \) and the red-shaded overestimate on the last page \( \to 0 \) too; so

\[
A = \lim_{n \to \infty} A(R^n) = \lim_{n \to \infty} \left( \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2} \right) = \frac{8}{3} \quad \text{is the exact area under the curve!}
\]

Limits of sums from areas

Let's go the other way around now. Suppose I hand you the sum

\[
S_n = \frac{1}{n} \sum_{i=1}^{n} \sqrt{1 - \frac{i^2}{n^2}}
\]

and suggest you find its limit. You might reason that this looks a bit like the last example, and wonder what happens if we think of \( \frac{1}{n} \sqrt{1 - \frac{i^2}{n^2}} \) as the area of a box under some curve. Which curve?
So $S_n$ is the area of the union of the rectangles displayed, and $\lim_{n \to \infty} S_n = \pi/4$, the area of a quarter of a disk of radius 1.

Estimating areas

A ball-park estimation of the area $A$ under a curve can be got by using upper and lower sums.

For example, $f(x) = 8^x \sin(\pi x/2)$ is a pretty weird function. You might think the area under its graph from $x = 0$ to $1$ is hard to estimate. But the function is increasing on $[0,1]$, and if we take $n=3$ (boxes) the upper and lower
Sums are a brave to compute:

\[
U = \frac{1}{3} \cdot f\left(\frac{1}{3}\right) + \frac{1}{3} \cdot f\left(\frac{2}{3}\right) + \frac{1}{3} \cdot f(1) \\
= \frac{1}{3} \left\{ 8^{\frac{1}{3}} \sin\left(\frac{\pi}{6}\right) + 8^{\frac{2}{3}} \sin\left(\frac{\pi}{3}\right) + 8^{1} \sin\left(\frac{\pi}{2}\right) \right\} \\
= \frac{1}{3} \left\{ 2 \cdot \frac{1}{2} + 4 \cdot \frac{\sqrt{3}}{2} + 8.1 \right\} = 3 + \frac{2\sqrt{3}}{3}
\]

\[
L = \frac{1}{3} \cdot f(0) + \frac{1}{3} \cdot f\left(\frac{1}{3}\right) + \frac{1}{3} \cdot f\left(\frac{2}{3}\right) = \frac{1}{3} \left\{ 3^0 + 8^{\frac{1}{3}} \frac{1}{2} + 8^{\frac{2}{3}} \frac{\sqrt{3}}{2} \right\} \\
= \frac{1}{3} + \frac{2\sqrt{3}}{3}, \quad \text{and so}
\]

\[1.48 \approx L \leq A \leq U \approx 4.15.\]

Pretty awful estimate, but taking \( n \) much larger leads to the idea of the definite integral we'll develop in the next lecture:

The difference between the areas of the red & green boxes (shown in yellow) will go \( \rightarrow 0 \) as we make more, \( \delta \) thinner, boxes \((n \rightarrow \infty)\). That is,
U gets smaller & L larger, in such a way that \( U - L \to 0 \).

Since "A" is sandwiched in between, BOTH U and L limit to A.

**Areas and distances**

A final conceptual remark is that the area under a curve is more than just a geometry problem. If \( y = f(t) \) is the speed of your car as a function of time, then the area under this curve from \( t = t_0 \) to \( t = t \), is the total distance traveled during that time. For example, if your speed (height of graph) is a constant \( C \) on \([t_0, t_1]\), then obviously distance \( \overline{t_1 - t_0} \cdot C \) is the area under \( C \).

For a curvy graph, the area over a small enough interval is approximated by a rectangle, and we
can take a limit as above to get the total distance exactly right. This connection will be useful later in understanding the Fundamental Theorem.

Likewise, if you are pushing a box along a straight path (which is smoother in some places than others) and \( y = F(x) \) is the force you applied as a function of position, then the area under the graph is the total work done. There are numerous other applications that we’ll see in due course.