Lecture 2: Area under a curve

When Gauss was 7, his teacher thought he had a good problem to keep the class busy. He said, find

$$1+2+3+\cdots+100 = \ ?$$

In a few seconds Gauss wrote 5,050. How did he do it?

$$1+2+3+\cdots+50+51+\cdots+98+99+100$$

50 pairs
each adds to 101
$50 \times 101 = 5{,}050.$

We can rewrite this sum more compactly in "sigma notation":

$$\sum_{i=1}^{100} i = 5050.$$

Can we also find $S_n = \sum_{i=1}^{n} i = 1+\cdots+n$ for any n?

First we need some rules about sums:

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

as long as these are the same, makes no difference whether its i, j, k, l etc. — called "dummy index".

SUM RULES

- $\sum_{k=1}^{n} 1 = \underbrace{1+\cdots+1}_{n \text{ times}} = n$
- $\sum_{k=1}^{n} (c \cdot a_k) = c \cdot \sum_{k=1}^{n} a_k$ (constant)
- $\sum_{k=1}^{n} (a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k$

$$\cancel{i^2} + 2i + 1 - \cancel{i^2}$$

Now then : $(i+1)^2 - i^2 = 2i + 1$, right ?

So $$\sum_{i=1}^{n} \left[(i+1)^2 - i^2 \right] = \sum_{i=1}^{n} (2i+1)$$

$$\sum_{i=1}^{n} (i+1)^2 - \sum_{i=1}^{n} i^2 = 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$$

$$\cancel{2^2 + 3^2 + \cdots + n^2} + (n+1)^2$$

$$-\left[1^2 + \cancel{2^2 + 3^2 + \cdots + n^2}\right]$$

collapsing sum!

$$\underbrace{(n+1)^2 - 1^2}_{n^2 + 2n + \cancel{1 - 1}} = 2S_n + n$$

$$n^2 + \cancel{2}n = 2S_n + \cancel{n}$$

$$\boxed{\frac{n(n+1)}{2}} = \frac{n^2+n}{2} = S_n = \sum_{i=1}^{n} i = \boxed{1 + \cdots + n}$$

You can do exactly the same thing, but with

$$(i+1)^3 - i^3 = 3i^2 + 3i + 1$$

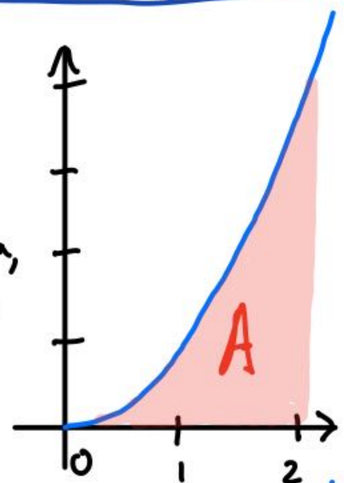
at the beginning, to find that

$$(*) \quad \boxed{\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}}$$

If you want the details, try it yourself and/or look in Appendix E of your book.

Areas from limits of sums

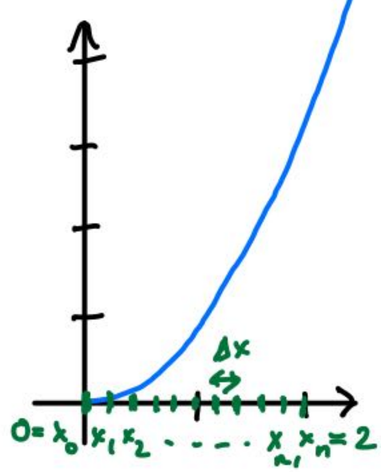
Let's use this formula (*) to find the area under a parabola, from 0 to 2:



We can't compute the area off the bat, so we'll approximate it with a polygon, and compute <u>that</u> area!

Begin by partitioning the interval $[0,2]$ into n subintervals of length

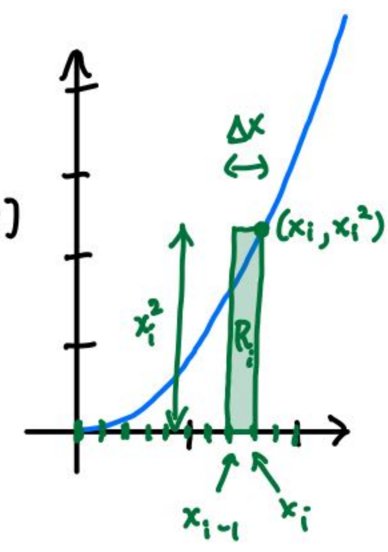
$$\Delta x = \frac{2}{n}.$$



So

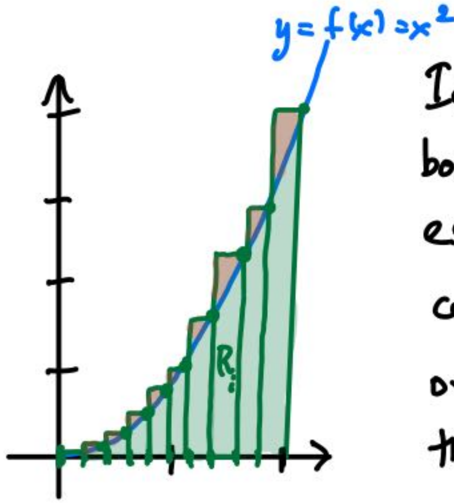
$$\begin{aligned} x_0 &= 0 \\ x_1 &= \frac{2}{n} \\ x_2 &= \frac{4}{n} \\ &\vdots \\ x_i &= \frac{2i}{n} \\ &\vdots \\ x_n &= \frac{2n}{n} = 2. \end{aligned}$$

Consider now the rectangle with base $[x_{i-1}, x_i]$ and height $f(x_i) = x_i^2$.



Its area is just

$$\begin{aligned} A(R_i) &= (\Delta x) \cdot x_i^2 \\ &= (\Delta x)\, f(x_i) \\ &= \frac{2}{n}\left(\frac{2i}{n}\right)^2 \\ &= \frac{8}{n^3}\, i^2. \end{aligned}$$



If we add the areas of these boxes up, we get a pretty good estimate of the area under the curve. This will be a slight overestimate, because we took the height of each box to be the largest value assumed by $f(x)$ on the i^{th} subinterval $[x_{i-1}, x_i]$.

[TERMINOLOGY: this is called an upper sum, and yields an upper bound on the area. You could also use lower sums (to get a lower bound), or always take the height to be $f(x_i)$ (right endpoint rule), $f(x_{i-1})$ (left endpoint rule), or $f\left(\frac{x_i + x_{i-1}}{2}\right)$ (midpoint rule), or just any point in $[x_{i-1}, x_i]$. Here the right endpoint rule & upper sum were the same, because $f(x)$ is increasing, but that won't always be the case!]

Let's do it! I'm going to add an extra index and call R_i instead R_i^n ← refers to width + # of boxes; ← refers to which box

and R^n will be the union of all these boxes as i runs from 1 to n. So

from last page

$$A(R^n) = \sum_{i=1}^{n} A(R_i^n) = \sum_{i=1}^{n} \frac{8}{n^3} i^2$$

area

$$= \frac{8}{n^3} \sum_{i=1}^{n} i^2 = \frac{8}{n^3} \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{\cancel{8}^{\,4}}{\cancel{6}_{\,3}} \frac{2n^3+3n^2+n}{n^3}$$

$$= \frac{4}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$

$$= \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2} .$$

As $n (= \#$ of boxes$) \to \infty$, $\Delta x \to 0$ and the red-shaded overestimate on the last page $\to 0$ too: so

$$A = \lim_{n\to\infty} A(R^n) = \lim_{n\to\infty}\left(\frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}\right)$$

$$= \frac{8}{3}$$ is the <u>exact</u> area under the curve!

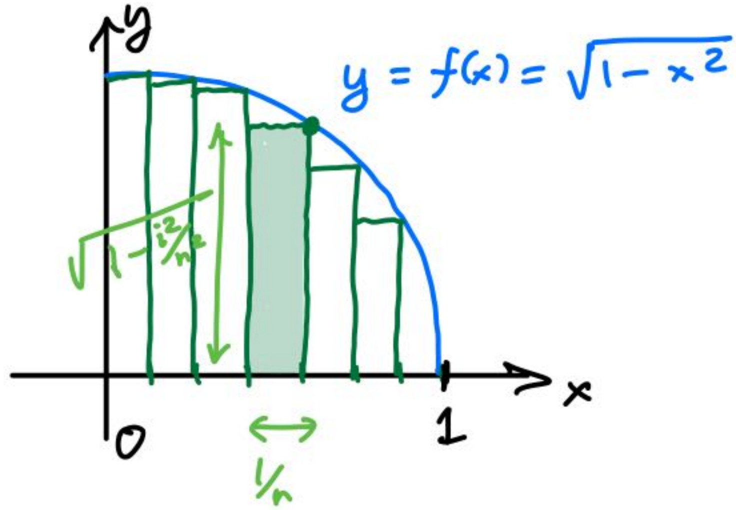
Limits of sums from areas

Let's go the other way around now. Suppose I hand you the sum

$$S_n = \frac{1}{n}\sum_{i=1}^{n} \sqrt{1 - i^2/n^2}$$

and suggest you find its limit. You <u>might</u> reason that this looks a bit like the last example, and wonder what happens if we think of $\frac{1}{n}\cdot\sqrt{1-\frac{i^2}{n^2}}$ as the area of a box under some curve. <u>Which</u> curve?

This one!



So S_n is the area of the union of the rectangles displayed, and $\lim_{n\to\infty} S_n = \pi/4$, the area of a quarter of a disk of radius 1.

Estimating areas

A ball-park estimation of the area A under a curve can be got by using upper and lower sums.

For example, $f(x) = 8^x \sin\left(\frac{\pi x}{2}\right)$ is a pretty weird function. You might think the area under its graph from $x = 0$ to 1 is hard to estimate. But the function is increasing on $[0,1]$, and if we take $n = 3$ (boxes) the upper and lower

Sums are a breeze to compute:

$$U = \frac{1}{3}\cdot f\left(\tfrac{1}{3}\right) + \frac{1}{3}\cdot f\left(\tfrac{2}{3}\right) + \frac{1}{3} f(1)$$

$$= \frac{1}{3}\left\{ 8^{1/3}\sin\left(\tfrac{\pi}{6}\right) + 8^{2/3}\sin\left(\tfrac{\pi}{3}\right) + 8^{1}\sin\left(\tfrac{\pi}{2}\right)\right\}$$

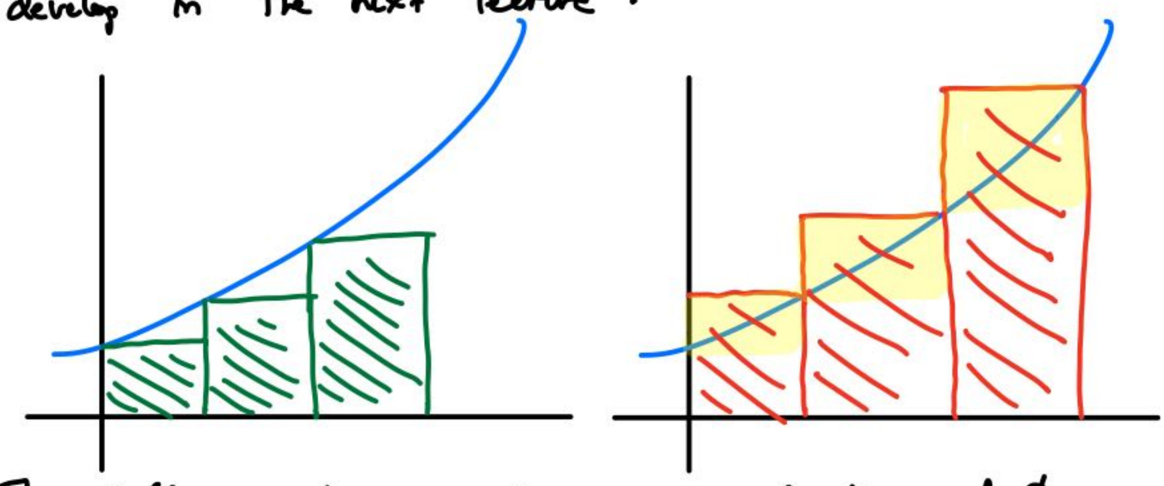
$$= \frac{1}{3}\left\{ \underbrace{2\cdot\tfrac{1}{2}}_{1} + \underbrace{4\cdot\tfrac{\sqrt{3}}{2}}_{2\sqrt{3}} + \underbrace{8\cdot 1}_{8} \right\} = 3 + \frac{2\sqrt{3}}{3}$$

$$L = \frac{1}{3} f(0) + \frac{1}{3} f\left(\tfrac{1}{3}\right) + \frac{1}{3} f\left(\tfrac{2}{3}\right) = \frac{1}{3}\left\{ 8^{0}\cdot 0 + 8^{1/3}\cdot\tfrac{1}{2} + 8^{2/3}\cdot\tfrac{\sqrt{3}}{2}\right\}$$

$$= \frac{1}{3} + \frac{2\sqrt{3}}{3},$$ and so

$$1.48 \approx L \le A \le U \approx 4.15.$$

Pretty awful estimate, but taking n much larger leads to the idea of the definite integral we'll develop in the next lecture:



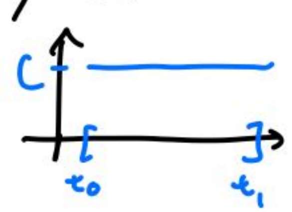
The difference between the areas of the red & green boxes (shown in yellow) will go $\longrightarrow 0$ as we make more, & thinner, boxes ($n \to \infty$). That is, U gets smaller & L larger, in such a way that

$$U - L \longrightarrow 0.$$

Since "A" is sandwiched in between, BOTH U and L limit to A.

Areas and distances

A final conceptual remark is that the area under a curve is more than just a geometry problem. If $y = f(t)$ is the speed of your car as a function of time, then the area under this curve from $t = t_0$ to $t = t_1$ is the total distance traveled during that time. For example, if your speed (height of graph) is a constant C on $[t_0, t_1]$, then obviously distance $= \underbrace{(t_1 - t_0)}_{\Delta t} \cdot \underbrace{C}_{\substack{f(t_1) \\ t_{\text{say}}}}$ is the area under



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For a curvy graph, the area over a small enough interval is approximated by a rectangle, and we can take a limit as above to get the total distance exactly right. This connection will be useful later in understanding the fundamental Theorem.

Likewise, if you are pushing a box along a straight path (which is smoother in some places than others) and $y = F(x)$ is the force you applied as a function of position, then the area under the graph is the total work done. There are numerous other applications that we'll see in due course.