Lecture 20 : Which integration method?

We've reached the end of our tour of integration methods, but so far the integrals have been carefully chosen to "fit" the method under discussion. When you encounter an integral in the wild, it may not be so clear which trick to pull out of your bag.

Ex 1/ Which method should you apply to

(i) $\displaystyle\int \frac{dx}{x^2-6x+25}$ (ii) $\displaystyle\int \frac{dx}{x^2-6x+8}$ (iii) $\displaystyle\int \frac{dx}{x^2-6x+9}$?

At first glance, they all look the same!

To see how they differ, apply the quadratic formula to (attempt to) factor the denominators:

(i) $x_\pm = \dfrac{6 \pm \sqrt{\underbrace{36-100}_{\text{negative}}}}{2}$ $\rightsquigarrow$ doesn't factor! (irreducible)

(ii) $x_\pm = \dfrac{6 \pm \sqrt{36-32}}{2} = \dfrac{6 \pm 2}{2} = \begin{cases} 4 \\ 2 \end{cases}$

$\rightsquigarrow$ factors as $(x-2)(x-4)$

(iii) $x_\pm = \dfrac{6 \pm \sqrt{36-36}}{2} = 3$ $\rightsquigarrow$ factors as $(x-3)^2$.

So (iii) is just $\displaystyle\int \frac{dx}{(x-3)^2} = \int (x-3)^{-2}dx = \frac{(x-3)^{-1}}{-1} + C = \frac{1}{3-x} + C$,

whereas (ii) will require using the method of partial fractions; while in (i), you'll <u>complete the square</u>:

(i) $x^2-6x+25 = x^2-6x+9+16 = (x-3)^2+16$

$$\Rightarrow \int \frac{dx}{x^2-6x+25} = \int \frac{dx}{(x-3)^2+4^2} = \int \frac{4\,du}{4^2u^2+4^2}$$

$$\left(\begin{array}{l} 4u = x-3 \\ 4\,du = dx \end{array}\right)$$

$$= \frac{1}{4}\int \frac{du}{u^2+1} = \frac{1}{4}\arctan(u) + C$$

$$= \frac{1}{4}\arctan\left(\frac{x-3}{4}\right) + C$$

(ii) $$\frac{1}{(x-2)(x-4)} = \frac{A}{x-2} + \frac{B}{x-4} = \frac{(A+B)x + (-2B-4A)}{(x-2)(x-4)}$$

$$\Rightarrow \begin{cases} A+B = 0 \\ -4A-2B = 1 \end{cases} \Rightarrow -2A = 1 \Rightarrow A = -\frac{1}{2} \Rightarrow B = \frac{1}{2};$$

so $$\int \frac{dx}{(x-2)(x-4)} = \frac{1}{2}\int \frac{dx}{x-4} - \frac{1}{2}\int \frac{dx}{x-2}$$

$$= \frac{1}{2}\ln|x-4| - \frac{1}{2}\ln|x-2| + C$$

$$= \frac{1}{2}\ln\left|\frac{x-4}{x-2}\right| + C.$$

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<u>Remark</u>: If in (ii) we had instead $\int \frac{x-3}{x^2-6x+8}\,dx$, partial fractions would work, but would be a waste of time:

substituting $\begin{cases} t = x^2-6x+8 \\ dt = 2x-6 \Rightarrow \frac{1}{2}dt = x-3 \end{cases}$ yields

$$\frac{1}{2}\int \frac{dt}{t} = \underset{(+C)}{\frac{1}{2}\ln|t|} = \underset{(+C)}{\frac{1}{2}\ln|x^2-6x+8|}.$$

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Here is a summary, briefer than the book's, of the methods we have learned and when to apply them:

① If the integrand is a rational function $\frac{P(x)}{Q(x)}$ (i.e. P, Q = polynomials):

- perform any cancellations (e.g. if P is a factor of Q)
- if $\deg P \geq \deg Q$, perform long division
- if $\deg P = \deg Q - 1$, it may be that $P(x) = \text{const} \times Q'(x)$; in that case, substitute $t = Q(x)$
- if Q is an irreducible quadratic and P is linear, write P as $Q' \times \text{const.} + \text{const.}$ Then deal with $\frac{\text{const.}}{Q(x)}$ by completing the square.
- if Q is reducible (and we don't simply have $\frac{\text{Const}}{(x-a)^n}$), use the method of partial fractions.

② If the integrand is $\cos^m x \sin^n x$ or $\tan^m x \sec^n x$, use the methods of Lectures 15–16 (Pythagorean identities, half-angle formulas, and occasionally integration by parts in the second case).

③ If the integrand contains a $\sqrt{x^2-a^2}$, $\sqrt{x^2+a^2}$, or $\sqrt{a^2-x^2}$, use a trigonometric substitution unless it's simply $\int \left(\sqrt{\pm x^2 \pm a^2}\right)^n x \, dx$ (in which case $t = \pm x^2 \pm a^2$ is better).

④ If the integrand is a product of two different kinds of functions, especially polynomial $\times$ {trig or e^x}, try integration by parts — unless there is an obvious substitution like in $\int x e^{x^2}\,dx$. Remember in choosing u & dv to prioritize for u those functions (ln, then arcsin/arctan, then polynomials) which "simplify" most under differentiation.

⑤ As a general rule, when a transcendental function (exp, trig, inv. trig, ln) or radical ($\sqrt[n]{\;}$) contains an argument more complicated than cx, make a substitution of $t =$ that argument. This is important even if the integral ends up being an integration by parts, trig substitution, or the like.

Ex 2/ $\int x^8 e^{x^3}\,dx = \frac{1}{3}\int t^2 e^t\,dt$ ← nasty argument!

$$\left(\left.\begin{array}{l} t = x^3 \\ dt = 3x^2\,dx \\ \frac{1}{3}dt = x^2\,dx \end{array}\right\} \Rightarrow \begin{array}{l} x^8\,dx = (x^3)^2 x^2\,dx \\ \quad = t^2 \cdot \frac{1}{3}\,dt \end{array}\right)$$

$$= \frac{1}{3}\left(t^2 e^t - 2\int t e^t\,dt\right) = \frac{1}{3}t^2 e^t - \frac{2}{3}\left(te^t - \int e^t\,dt\right)$$

$$\left(\begin{array}{ll} \multicolumn{2}{l}{\underline{\text{Now}}\text{ use } \int\text{-by-parts:}} \\ u = t^2 & dv = e^t\,dt \\ du = 2t\,dt & v = e^t \end{array}\right) \quad \left(\begin{array}{ll} u = t & dv = e^t\,dt \\ du = dt & v = e^t \end{array}\right) \quad = \frac{1}{3}(t^2 - 2t + 2)e^t + C$$

Ex 3/ Here is another trio of similar-looking integrands:

(i) $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx$, (ii) $\displaystyle\int \frac{dx}{x^3\sqrt{1-x^2}}$, (iii) $\displaystyle\int \frac{\sqrt{1-x^2}}{x}\,dx$.

(any guesses?)

__(i)__ is straightforward substitution: $t = 1-x^2$, $dt = -2x\,dx$, $-\frac{1}{2}dt = x\,dx$ makes it into

$$\int \frac{-\frac{1}{2}\,dt}{t^{1/2}} = -\frac{1}{2}\int t^{-1/2}\,dt = \underset{(+C)}{-\frac{1}{2}\,\frac{t^{1/2}}{1/2}} = \underset{(+C)}{-t^{1/2}} = -\sqrt{1-x^2} + C$$

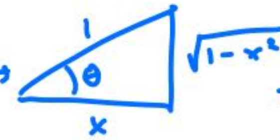
__(ii)__ is trig substitution: $x = \cos\theta$ (or $\sin\theta$), $dx = -\sin\theta\,d\theta$ turns it into

$$\int \frac{-\sin\theta\,d\theta}{\cos^3\theta\,\sqrt{1-\cos^2\theta}} = -\int \frac{\cancel{\sin\theta}\,d\theta}{\cos^3\theta\,\cancel{\sin\theta}} = -\int \sec^3\theta\,d\theta$$

$$= -\frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta+\tan\theta| + C$$

(See example at end of Lect. 16)

Now use $x = \cos\theta \longleftrightarrow$



$\Longrightarrow$ $\sec\theta = \frac{1}{x}$, $\tan\theta = \frac{\sqrt{1-x^2}}{x}$

$$= -\frac{\sqrt{1-x^2}}{2x^2} - \frac{1}{2}\ln\left|\frac{1+\sqrt{1-x^2}}{x}\right| + C.$$

__(iii)__ could try trig substitution: $\begin{pmatrix} x = \sin\theta \\ dx = \cos\theta\,d\theta \end{pmatrix}$

$$= \int \frac{\sqrt{1-\sin^2\theta}}{\sin\theta}\cos\theta\,d\theta$$

$$= \int \frac{\cos^2\theta}{\sin\theta}\,d\theta$$ ... at which point we're stuck.

Try instead: $\left.\begin{aligned} t &= \sqrt{1-x^2} \\ t^2 &= 1-x^2 \\ -t\,dt &= x\,dx \end{aligned}\right\}$ yields $\displaystyle\int \frac{\sqrt{1-x^2}}{x}\,dx = \int \frac{\sqrt{1-x^2}}{x^2}\,x\,dx$

$$= -\int \frac{t}{1-t^2}\,t\,dt = \int \frac{t^2}{t^2-1}\,dt$$

$$= \int \frac{t^2-1}{t^2-1}\,dt + \int \frac{dt}{t^2-1}$$

easy partial fractions: $\displaystyle\frac{1}{(t-1)(t+1)} = \frac{1/2}{t-1} - \frac{1/2}{t+1}$

$$= \int dt + \frac{1}{2}\int \frac{dt}{t-1} - \frac{1}{2}\int \frac{dt}{t+1}$$

$$= t + \frac{1}{2}\ln|t-1| - \frac{1}{2}\ln|t+1| + C$$

$$= t + \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C$$

$$= \sqrt{1-x^2} + \frac{1}{2}\ln\left|\frac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}+1}\right| + C.$$

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