Lecture 21 : Exam II Review

Format : 10 multiple choice problems

2 handgraded (multipart) problems

worth 15% of grade

Material covered : §§5.3-5, §§6.1-5, §§7.1-5.

(I) Fundamental Theorem of Calculus

(II) Methods of Integration

(III) Applications of Integration

By (I), I mean :

(a) knowing how to apply the net change theorem to compute integrals

(b) using FTC v.1 to compute derivatives like this :

Ex/ Find $h'(4)$ if $h(x) = \int_1^{\sqrt{x}} \frac{z^2}{z^4+1}\,dz$.

We have $h'(x) = \frac{d}{dx}\sqrt{x} \cdot \frac{(\sqrt{x})^2}{(\sqrt{x})^4+1} = \frac{1}{2\sqrt{x}} \cdot \frac{x}{x^2+1}$

$\Rightarrow h'(4) = \frac{1}{2 \cdot 2} \cdot \frac{4}{16+1} = \frac{1}{17}$. //

Turning to (II), we have the following techniques:

- antidifferentiation
- substitution ($t = g(x)$, $dt = g'(x)\,dx$) — e.g. $f'(x)f(x)^{\alpha}$, $f'(x)e^{f(x)}$
- integration by parts ($\int u\,dv = uv - \int v\,du$)
- $\frac{1}{2}$-angle & pythagorean formulas ("trig integrals") — exam only involves $\sin^a \cos^b$
- trigonometric substitution ($x = \sin/\cos/\tan/\sec(\theta)$) ← exam will only use these
- partial fractions (exam will only involve $(x-a)(x-b)$ denominators)

Here are some specific integrals I think you should be know how to work:

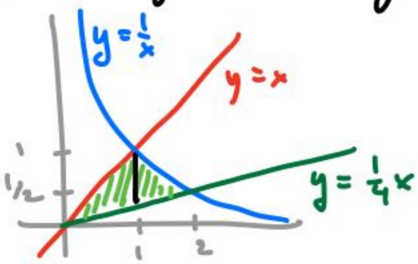
- antiderivatives: x^k, $\frac{1}{x^k}$, $\sqrt{x}$, $\frac{1}{\sqrt{x}}$, $\sin/\cos/e^x$, $\frac{1}{1+x^2}$, $\frac{1}{\sqrt{1-x^2}}$
- $\int_0^1 \sqrt{4-3x^2}\,dx$ trig subst.
- $\int x\sqrt{a^2-x^2}\,dx$ substitution
- $\int \ln(a-x)\,dx$ $\int$ by parts
- $\int \sin^8 x \cos^3 x\,dx$ Pythag.
- $\int \sin^8 x \cos^4 x\,dx$ $\frac{1}{2}$-angle (though this is uglier than anything you'd see on the exam)
- $\int x^2 \sin(x)\,dx$ $\int$ by parts (twice)
- $\int_5^6 \frac{dx}{x^2-9x+14}$ partial fractions
- $\int_3^7 \frac{dx}{x^2-6x+25}$ complete the square

If you are comfortable with these, you'll be fine.

(III) Applications :

- average value of a function
- areas between curves

Ex/ Find the area of the region bounded by $y=\frac{1}{x}$, $y=x$, $y=\frac{1}{4}x$, with $x>0$.



$$A = \int_0^1 \left(x-\frac{1}{4}x\right)dx + \int_1^2\left(\frac{1}{x}-\frac{1}{4}x\right)dx$$

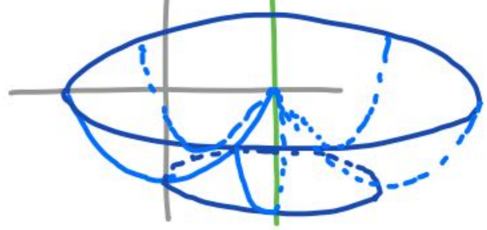
$$= \frac{x^2}{2}\Big|_0^1 - \frac{x^2}{8}\Big|_0^1 + \ln x\Big|_1^2 - \frac{x^2}{8}\Big|_1^2 = \cancel{\frac{1}{2}} - \cancel{\frac{1}{8}} + \ln 2 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{8}}$$

$$= \ln(2)$$

//

- volume of solids of revolution — shells & washers/disks (that's all that will be on exam)

Ex/ Find the volume of the solid obtained by revolving the region bounded by $y=x^2-1$ and $y=0$ about the line $x=1$.



shells: $V = \int_{-1}^{1} 2\pi\underbrace{(1-x)}_{r(x)}\underbrace{(1-x^2)}_{h(x)}\,dx = \frac{8\pi}{3}$

"

washers: $V = \int_{-1}^{0}\pi\left(\big(\underbrace{1+\sqrt{1+y}}_{R(y)}\big)^2-\big(\underbrace{1-\sqrt{1+y}}_{r(y)}\big)^2\right]dy$

//

- work (exam will only cover the type of problem in which rope is being hauled up or water pumped out of a vessel)

Ex / Suppose the "dish" from last example is filled with water, which must be pumped to height $y=0$. Find the work done:

Cut up the water into washers of $\begin{cases}\text{area } A(y) = \pi\left(R(y)^2 - r(y)^2\right), \\ \text{thickness } dy\end{cases}$ ← assume units = meters

as in last example. Then this slice weighs $A(y)\,dy \times 1000\ \text{kg}/\text{m}^3 \times \underbrace{10\ \text{m}/\text{s}^2}_{\text{force to counteract gravity}}$

$$\Rightarrow W = \int_{-1}^{0} 10^4 A(y)(-y)\,dy$$

$$= -10^4\pi \int_{-1}^{0} 4y\sqrt{1+y}\,dy \underset{\substack{\uparrow \\ t = 1+y}}{=} -10^4\pi \int_0^1 4(t-1)t^{1/2}\,dt$$

$$= 4\pi\cdot 10^4 \int_0^1 \left(t^{1/2} - t^{3/2}\right)dt = 4\pi\cdot 10^4 \cdot \frac{4}{15} = \frac{16}{15}\pi \times 10^4\ \text{N}. \quad //$$

- linear motion

Ex / I am applying a constant force of 100 N to a 100 kg block of ice which is melting at 1 kg/s. Determine the velocity as a function of time, starting from rest at $t=0$.

Use $F = ma$. More precisely, $100\ \text{N} = F = m(t)\,a(t)$,

with $m(t) = 100 - t \implies a(t) = \dfrac{100}{100-t} \implies$

$$v(t) = \int a(t)\,dt = -100\ln(100-t) + C$$

$$0 = v(0) = -100\ln 100 + C \implies C = 100\ln 100$$

$$\implies v(t) = 100\ln\left(\frac{100}{100-t}\right) = -100\ln\left(1 - t/100\right). \quad //$$

Good luck!