Lecture 22: Improper Integrals

In applications of integration, one frequently has to break the rule of integrating a bounded, piecewise-continuous function over a finite interval $[a,b]$:

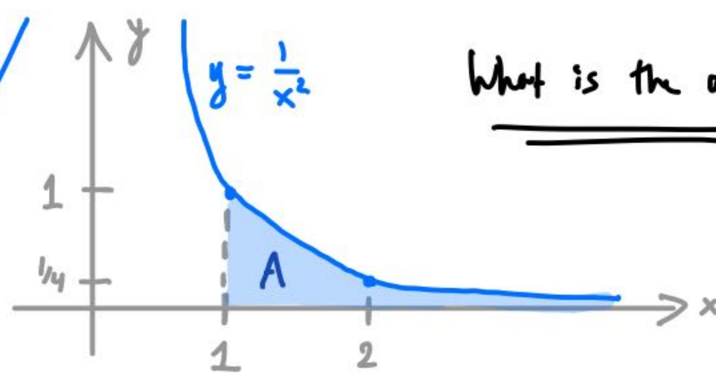
① <u>Unbounded intervals of integration</u> arise in (for example)

- physics (electromagnetic potentials, escape velocity from a planet)
- probability & statistics (normal distribution a.k.a. "bell curve")
- electrical engineering (Laplace & Fourier transforms)

② <u>Functions unbounded on the interval</u> of integration arise whenever one has a "singularity" (like a point electric charge or black hole), and typically involve vertical asymptotes.

<u>Unbounded Intervals</u>

Ex 1/



<u>What is the area A?</u>

definition!

$$A = \text{"}\int_1^\infty \frac{1}{x^2}\,dx\text{"} := \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx$$

$$= \lim_{b\to\infty}\left[\frac{-1}{x}\right]_1^b = \lim_{b\to\infty}\left(-\frac{1}{b}+1\right)$$

$$= 1.$$ This integral "converges". //

<u>Definitions</u>:

$$\int_a^\infty f(x)\,dx := \lim_{b\to\infty}\int_a^b f(x)\,dx$$

$$\int_{-\infty}^b f(x)\,dx := \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$

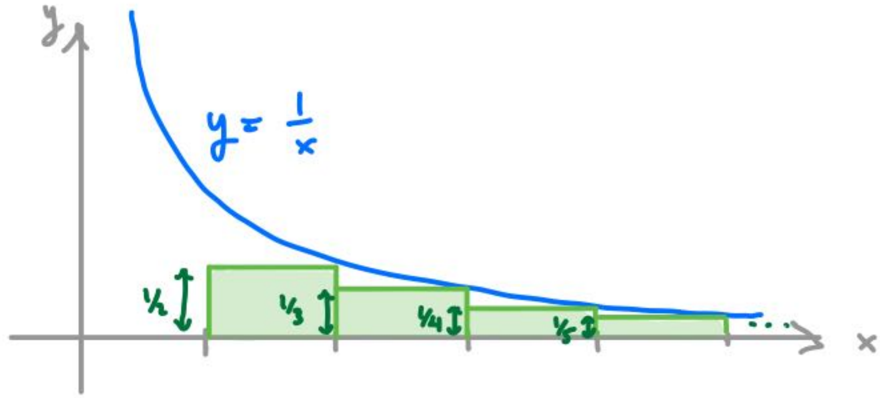
Ex 2/ $\int_1^\infty \frac{1}{x}\,dx$, which doesn't look too different...

$$= \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty}\left[\ln|x|\right]_1^b = \lim_{b\to\infty} \ln b - \cancelto{0}{\ln 1}$$

$$= \infty.$$

That is, <u>this</u> integral "diverges". //

This is related to the divergence of the (nominally smaller) sum of areas of these boxes:



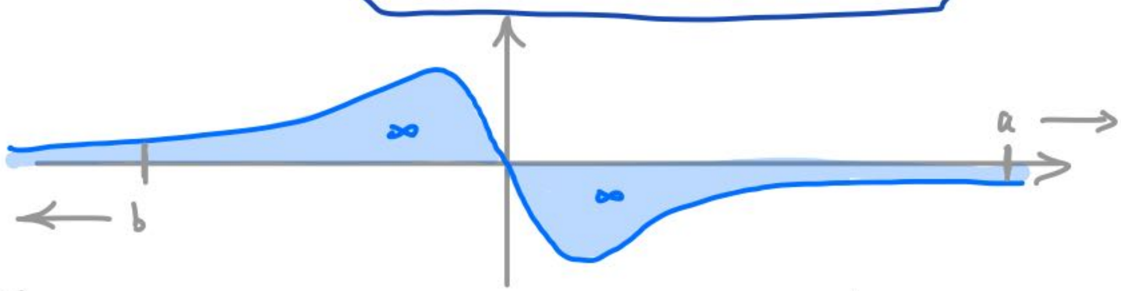
$$A(\text{boxes}) = \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{> \frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{> \frac{1}{2}} + \cdots$$

$$> \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = \infty.$$

You can use sums to show integrals converge or diverge in this way. More on this after Spring Break.

One other type I haven't yet mentioned:

$$\int_{-\infty}^{\infty} f(x)\,dx := \boxed{\int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx} \quad (*)$$



The convention is that both terms of $(*)$ <u>must converge separately</u>: in a scenario like that pictured, the two pieces don't cancel to give zero if the areas are both infinite, because one could take $a \to \infty$ and $b \to -\infty$ at different rates and get any value as the limit. So in the picture, we'd just say $\int_{-\infty}^{\infty} f(x)\,dx$ <u>diverges</u>.

$\boxed{\text{Try}}$ $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{4+x^2}$

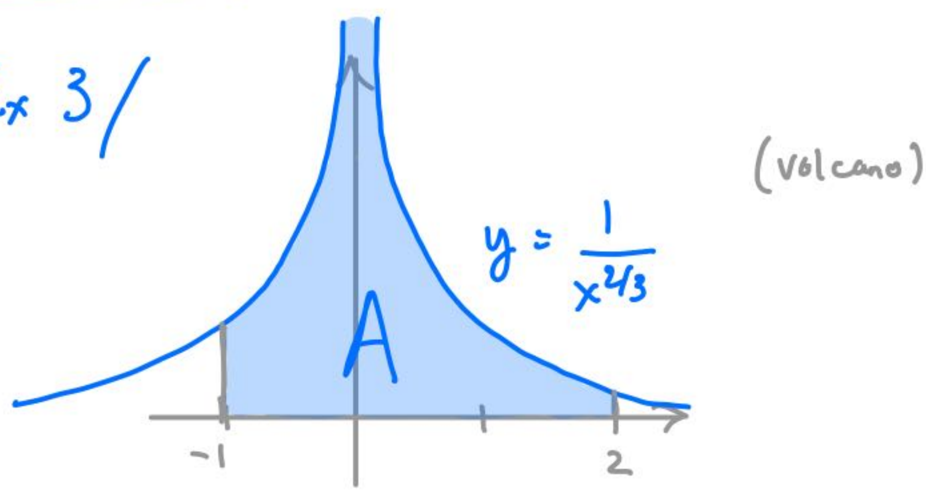
<u>Ans</u>: $\displaystyle\int_0^{\infty} \frac{dx}{4+x^2} + \int_{-\infty}^{0} \frac{dx}{4+x^2} = \lim_{b\to\infty}\int_0^b \frac{dx}{4+x^2} + \lim_{a\to-\infty}\int_a^0 \frac{dx}{4+x^2}$

$\displaystyle = \lim_{b\to\infty}\left[\tfrac{1}{2}\arctan\left(\tfrac{x}{2}\right)\right]_0^b + \lim_{a\to-\infty}\left[\tfrac{1}{2}\arctan\left(\tfrac{x}{2}\right)\right]_a^0$

$\displaystyle = \lim_{b\to\infty}\tfrac{1}{2}\arctan\left(\tfrac{b}{2}\right) - \lim_{a\to-\infty}\tfrac{1}{2}\arctan\left(\tfrac{a}{2}\right) = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}.$

Infinite integrands

Ex 3/



$$A = \int_{-1}^{2} \frac{dx}{x^{2/3}} := \underbrace{\int_{-1}^{0} \frac{dx}{x^{2/3}}}_{} + \underbrace{\int_{0}^{2} \frac{dx}{x^{2/3}}}_{}$$

same idea: both must exist seperately!

$$:= \lim_{b \to 0^-} \underbrace{\int_{-1}^{b} \frac{dx}{x^{2/3}}}_{} + \lim_{a \to 0^+} \underbrace{\int_{a}^{2} \frac{dx}{x^{2/3}}}_{}$$

integrals of continuous functions that we can already compute

$$= \lim_{b \to 0^-} \left[\frac{x^{1/3}}{1/3} \right]_{-1}^{b} + \lim_{a \to 0^+} \left[\frac{x^{1/3}}{1/3} \right]_{a}^{2}$$

$$= \lim_{b \to 0^-} \left(3 \underbrace{b^{1/3}}_{\to 0} + 3 \right) + \lim_{a \to 0^+} \left(3 \cdot 2^{1/3} - 3 \underbrace{a^{1/3}}_{\to 0} \right)$$

$$= 3 + 3 \cdot 2^{1/3} = 3\left(1 + \sqrt[3]{2}\right). \quad //$$

$\boxed{\text{Try}}$ $\displaystyle \int_0^3 \frac{dx}{(x-1)^3}$

Ans: $\displaystyle \lim_{a \to 1^-} \int_0^a \frac{dx}{(x-1)^3} + \lim_{b \to 1^+} \int_b^3 \frac{dx}{(x-1)^3} =$

$$\lim_{a \to 1^-} \underbrace{\left[\frac{-1/2}{(x-1)^2} \right]_0^a}_{\downarrow \, -\infty} + \lim_{b \to 1^+} \underbrace{\left[\frac{-1/2}{(x-1)^2} \right]_b^3}_{\downarrow \, +\infty}$$

... again, you can't cancel ∞'s: Diverges!