Lecture 23 : More Improper Integrals

Recall that if f is continuous on $(a,b]$ then we define $\int_a^b f(x)\,dx := \lim_{A\to a^+}\int_A^b f(x)\,dx$ if this limit exists ; if it's continuous on $[a,b)$ then use $\lim_{B\to b^-}\int_a^B f(x)\,dx$; if it's continuous on $[a,b]$ <u>except</u> at c, then we define $\int_a^b = \int_a^c + \int_c^b$.

Likewise, if f is continuous on $[a,\infty)$, then by $\int_a^\infty f(x)\,dx$ we mean $\lim_{b\to\infty}\int_a^b f(x)\,dx$, if this exists ; and so on.

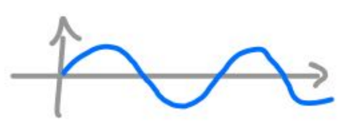
Today we continue with examples and then do some applications.

Ex 1/ $\displaystyle\int_0^\infty \sin(x)\,dx = \lim_{b\to\infty}\int_0^b \sin(x)\,dx$

not infinite, but keeps oscillating back & forth forever

$$= \lim_{b\to\infty}\Big[-\cos(x)\Big]_0^b = \lim_{b\to\infty}\big(1-\cos(b)\big) \quad \underline{\underline{\text{DNE}}}$$

i.e. this integral is divergent



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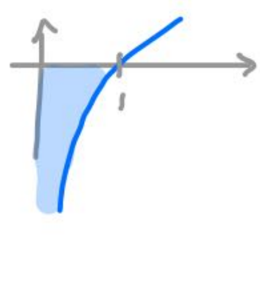
Ex 2/ $\displaystyle\int_0^1 \ln(x)\,dx = \lim_{a\to 0^+}\int_a^1 \ln(x)\,dx$

$$= \lim_{a\to 0^+}\left(x\ln x\Big|_a^1 - \int_a^1 \cancel{x}\frac{dx}{\cancel{x}}\right) = \lim_{a\to 0^+}\left(-a\ln a - 1 + a\right)$$

$\begin{pmatrix} u=\ln(x) & dv=dx \\ du=\frac{dx}{x} & v=x \end{pmatrix}$

$$= -1$$



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Ex 3 / Let $\alpha > 0$, and consider (if $\alpha \neq 1$)

$$\boxed{\int_0^1 \frac{dx}{x^\alpha}} = \lim_{A\to 0^+} \int_A^1 \frac{dx}{x^\alpha} = \lim_{A \to 0^+} \left[\frac{x^{1-\alpha}}{1-\alpha} \right]_A^1$$

$$= \lim_{A\to 0^+} \left(\frac{1}{1-\alpha} - \frac{A^{1-\alpha}}{1-\alpha} \right) = \boxed{\begin{cases} \frac{1}{1-\alpha}, & \alpha < 1 \\ \infty, & \alpha > 1. \end{cases}}$$

If $\alpha = 1$, $\int_0^1 \frac{dx}{x} = \lim_{A\to 0^+} (-\ln A) = \infty$.

On the other hand, for $\alpha \neq 1$

$$\boxed{\int_1^\infty \frac{dx}{x^\alpha}} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^\alpha} = \lim_{b\to\infty} \left[\frac{x^{1-\alpha}}{1-\alpha} \right]_1^b = \lim_{b\to\infty} \left(\frac{b^{1-\alpha}}{1-\alpha} - \frac{1}{1-\alpha} \right)$$

$$= \boxed{\begin{cases} \infty, & \alpha < 1 \\ \frac{1}{\alpha - 1}, & \alpha > 1. \end{cases}}$$

Again, for $\alpha = 1$ the integral diverges.

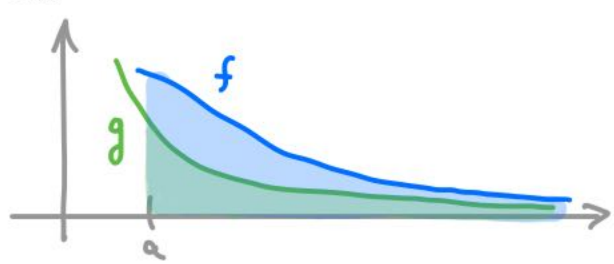
These will be useful for comparing with infinite sums later on. //

Speaking of comparison, we have the following fact:

if $0 \leq g \leq f$ on $[a, \infty)$, then

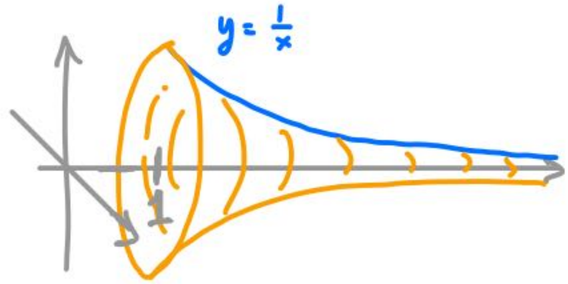
- $\int_a^\infty g\, dx$ diverges $\Longrightarrow$ $\int_a^\infty f\, dx$ does too
- $\int_a^\infty f\, dx$ converges $\Longrightarrow$ $\int_a^\infty g\, dx$ does too.

This makes sense if you view it in terms of areas under f & g:



Applications

Ex 4 (Gabriel's Horn)



To find the volume of the horn, we take vertical slices (disks) to get

$$V = \int_1^\infty A(x)\,dx = \int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx$$

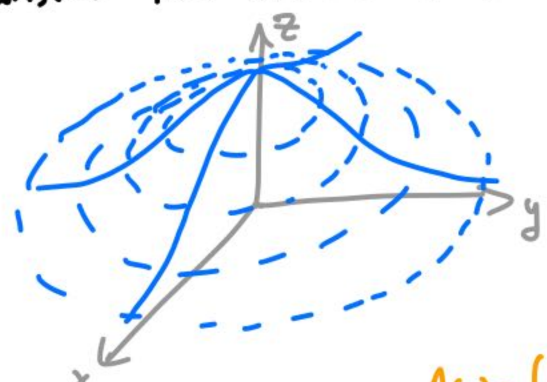
$$= \pi \cdot \lim_{b\to\infty} \int_1^b \frac{dx}{x^2} = \pi \cdot \frac{1}{2-1} = \pi \quad //$$

(Ex 3)

Ex 5 (the bell curve) Let $I := \displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx$.

To find this, consider the volume of the solid of revolution described by

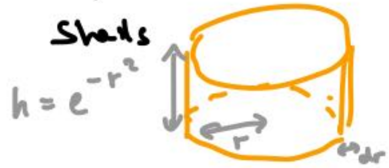
$$z = e^{-r^2} = e^{-(x^2+y^2)}$$
$$= e^{-x^2}\cdot e^{-y^2}.$$



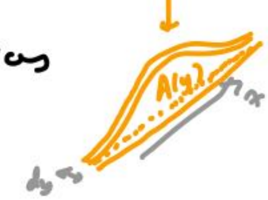
$$A(y) = \int_{-\infty}^{\infty} e^{-y^2}\cdot e^{-x^2}\,dx = e^{-y^2}\cdot I$$

We can compute this volume by

Shells



OR vertical slices



- Shells yield $V = 2\pi \int_0^\infty r e^{-r^2}\, dr = \pi \int_0^\infty e^{-u}\, du$

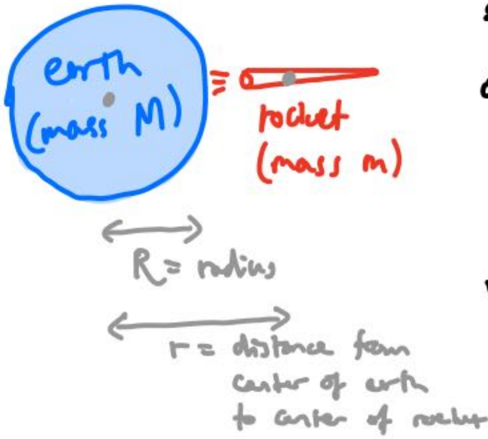
$$= \pi \cdot \lim_{b\to\infty} \int_0^b e^{-u}\, du = \pi \cdot \lim_{b\to\infty} \left[-e^{-u}\right]_0^b$$

$$= \pi \cdot \lim_{b\to\infty} \left(1 - e^{-b}\right) = \pi .$$

- Slices yield $V = \int_{-\infty}^{\infty} A(y)\, dy = I \cdot \int_{-\infty}^{\infty} e^{-y^2}\, dy = I^2 .$

So $I^2 = \pi \Rightarrow I = \sqrt{\pi}$. //

Ex 6 (Escape velocity)



Newton's Law of Gravitation says the force of gravity between objects shown is

$$F = G \frac{mM}{r^2} ,$$

where $G = 6.674 \times 10^{-11}\ m^3/kg \cdot s^2$

(eg for earth, $R = 6.371 \times 10^3$ km
$M = 5.972 \times 10^{24}$ kg)

The work required to move the rocket from the surface of the earth to ∞ is

$$W = \int_R^\infty G \frac{mM}{r^2}\, dr = \lim_{b\to\infty} \left[-m \frac{GM}{r}\right]_R^b = m \frac{GM}{R} .$$

The kinetic energy "available" to do this work when the rocket has an initial speed of v_0 , is $\frac{1}{2} m v_0^2$. So the rocket will escape earth's gravity $\Leftrightarrow$

$$\frac{1}{2}\cancel{m}v_0^2 \geq \cancel{m}\frac{GM}{R} \iff$$

$$v_0 \geq \sqrt{\frac{2GM}{R}} \approx 11.18 \text{ km/s},$$

which is thus called the escape velocity. //