Lecture 24: Arc length of a curve

Ex 11/ You are designing a slide for the City Museum in the shape of a parabola:

\[ y = x^2 \ (= f(x)) \]

(in meters)

and need to know whether your 5 m sheet of metal will suffice. Having taken a Calculus course, you decide to chop the graph up into little bits and estimate the length of each bit:

The slope of the \( i \)th secant segment is given by \( f(x_i^*) \), for some \( x_i^* \in [x_{i-1}, x_i] \). So its height is \( f(x_i^*) \Delta x \).
By Pythagorus, the green segment has length

\[(\Delta s)_i = \sqrt{\Delta x^2 + (f'(x_i^*) \Delta x)^2} = \sqrt{1 + (f'(x_i^*))^2 \cdot (\Delta x)}.\]

So we get

\[L = \sum_{i=1}^{n} (\Delta s)_i = \sum_{i=1}^{\infty} \sqrt{1 + (f'(x_i^*))^2} \cdot (\Delta x)\]

\[\lim_{n \to \infty} L = \int_{0}^{2} \sqrt{1 + (f'(x))^2} \, dx\]

\[f'(x) = 2x \int_{0}^{2} \sqrt{1 + 4x^2} \, dx\]

\[= \int_{0}^{\text{arctan}(4)} \sqrt{1 + \tan^2 \theta} \cdot \frac{1}{2} \sec^2 \theta \, d\theta\]

\[= \frac{1}{2} \int_{0}^{\text{arctan}(4)} \sec^3 \theta \, d\theta\]

\[= \frac{1}{4} \left[ \sec(\theta) \tan(\theta) + \ln |\sec(\theta) + \tan(\theta)| \right]_{\text{arctan}(4)}^{0}\]

\[= \frac{1}{4} \sqrt{1 + 4} + \frac{1}{4} \ln |\sqrt{1 + 4} + 4|\]

\[= \sqrt{1 + 4} + \frac{1}{4} \ln (\sqrt{1 + 4})\]

\[\approx 4.8356.\]

A general formula for the arclength of \(y = f(x)\) from \(x = a\) to \(b\) emerges from this first example:

\[L = \int_{a}^{b} \sqrt{1 + (f'(x))^2} \, dx \quad \text{or} \quad \int_{a}^{b} \sqrt{1 + (\frac{dy}{dx})^2} \, dx\]
One can also "flip this around" if \( x \) is given as a function of \( y \):

\[
L = \int_{\alpha}^{\beta} \sqrt{1 + (g'(y))^2} \, dy
\]

\[
= \int_{\alpha}^{\beta} \sqrt{\frac{2}{g(y)^2}} \, dy
\]

Notice that both formulas suggest an integral of
\[
\sqrt{(dx)^2 + (dy)^2},
\]
which is sometimes denoted "ds" for "differential of arc length". (The "s" comes from Latin: "Spatium").

The second thing you see from the example is that we get a complicated integral from a "nice" function. Sometimes "na"tive" functions give nicer answers.

**Try:** Find the length of \( x = \sqrt{\frac{y^2}{4} - \frac{1}{2} \ln y} \)

from \( y = 1 \) to \( y = \sqrt{e} \).

\[
\text{Ans.: } L = \int_{1}^{\sqrt{e}} \sqrt{1 + (g'(y))^2} \, dy = \int_{1}^{\sqrt{e}} \sqrt{1 + \left(\frac{y}{2} - \frac{1}{2y}\right)^2} \, dy
\]

\[
= \int_{1}^{\sqrt{e}} \sqrt{\frac{y^2}{4} + 1 - 1 + \frac{1}{4y^2}} \, dy = \int_{1}^{\sqrt{e}} \sqrt{\left(\frac{y}{2} + \frac{1}{2y}\right)^2} \, dy
\]
Ex 2/ Of what function is a freely hanging wire/chain (with only the two ends fixed) the graph?

You may have heard of a “catenary”, and know that the Arch is a flattened Catenary, but what sort of curve is that?

- \( O = \text{sum of horizontal forces} \)
  \[ = T_x \cos \theta_x - T_0 \quad \Rightarrow \quad T_x \cos \theta_x = T_0 \]

- \( O = \text{sum of vertical forces} \)
  \[ = T_x \sin \theta_x - g \delta s(x) \quad \Rightarrow \quad T_x \sin \theta_x = g \delta s(x) \]
Dinding the boxed equation gives \( \tan \theta_x = \frac{g \sqrt{r}}{T_0} s(x) \), and \( \tan \theta_x \) is either all just \( f'(x) \).

So we have

\[
\frac{d}{dx} a \cdot s(x) = a \int_0^x \sqrt{1 + (f'(x))^2} \, dx
\]

\[
f''(x) = a \sqrt{1 + (f'(x))^2}
\]

or

\[
(f''(x))^2 = a^2 + a^2 (f'(x))^2.
\]

You may be familiar with the facts that

\[
(Sinh(x))^2 + 1 = (Cosh(x))^2 \quad \text{and} \quad \begin{cases} \text{Cosh}(x) = \text{Sinh}(x) \\ \text{Sinh}(x) = \text{Cosh}(x). \end{cases}
\]

These imply at once that

\[
f(x) = \frac{1}{a} \cosh(ax)
\]

solves \((\star)\), and is (up to \(+ C\)) the unique solution.

(The Buckmeyer Arch is about \( \frac{2}{3} \) \( x \) such an \( f \), flipped over.)

Try: Find the "arc-length function" \( s(x) \) for the curve \( y = \frac{2}{3} x^{3/2} \), i.e., the length of the curve from \((0,0)\) to \((x, \frac{2}{3} x^{3/2})\).

\[
\text{Ans: } s(x) = \int_0^x \sqrt{1 + (f'(t))^2} \, dt = \int_0^x \sqrt{1 + (t^{3/2})^2} \, dt
\]

\[
= \int_0^x \sqrt{1 + t} \, dt = \left[ \frac{2}{3} (1 + x^{3/2}) \right]_0^x = \frac{2}{3} \left( (1 + x^3/2) - 1 \right).
\]