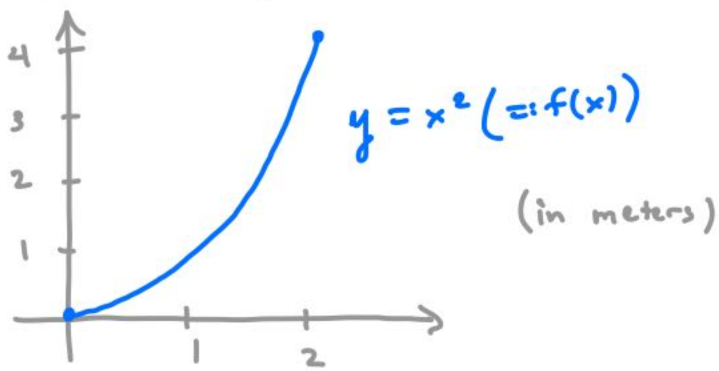
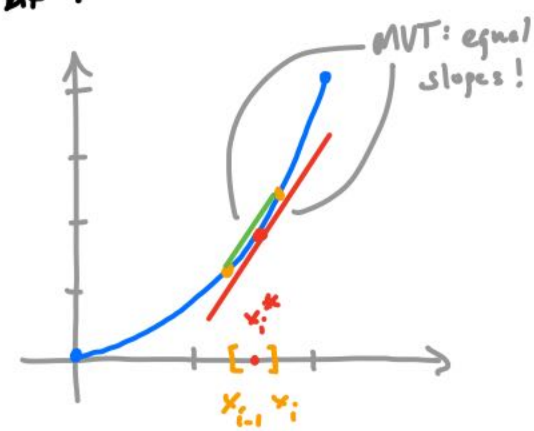
Lecture 24: Arclength of a curve

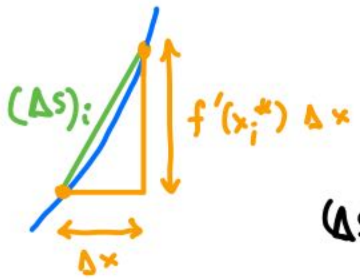
Ex 1 // You are designing a slide for the City Museum in the shape of a parabola



and need to know whether your 5 m sheet of metal will suffice. Having taken a Calculus course, you decide to chop the graph up into little bits and estimate the length of each bit:



The slope of the i^{th} secant segment is given by $f'(x_i^*)$, for some $x_i^* \in [x_{i-1}, x_i]$. So its height is $f'(x_i^*)\Delta x$.



By Pythagoras, the green segment has length

$$(\Delta s)_i = \sqrt{(\Delta x)^2 + (f'(x_i^*)\,\Delta x)^2} = \sqrt{1+(f'(x_i^*))^2}\cdot(\Delta x).$$

So we get $L \approx \sum_{i=1}^{n} (\Delta s)_i = \sum_{i=1}^{n} \sqrt{1+(f'(x_i^*))^2}\,(\Delta x)$

$$\Big\downarrow \lim_{n\to\infty}$$

$$L = \int_0^2 \sqrt{1+(f'(x))^2}\,dx$$

$f'(x) = 2x$

$$= \int_0^2 \sqrt{1+4x^2}\,dx$$

$\frac{s^2}{c^2} + \frac{c^2}{c^2} = 1/c^2$

$t^2+1 = s^2$

$4x^2 = \tan^2\theta$

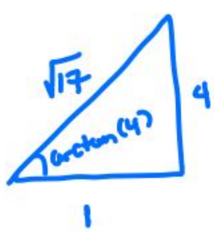
$2x = \tan\theta$

$2\,dx = \sec^2\theta\,d\theta$

$$= \int_0^{\arctan(4)} \sqrt{1+\tan^2\theta}\cdot\frac{1}{2}\sec^2\theta\,d\theta$$

$$= \frac{1}{2}\int_0^{\arctan(4)} \sec^3\theta\,d\theta$$

$$= \frac{1}{4}\Big[\sec(\theta)\tan(\theta) + \ln|\sec(\theta)+\tan(\theta)|\Big]_0^{\arctan(4)}$$



$$= \frac{1}{4}\sqrt{17}\cdot 4 + \frac{1}{4}\ln\left|\sqrt{17}+4\right|$$

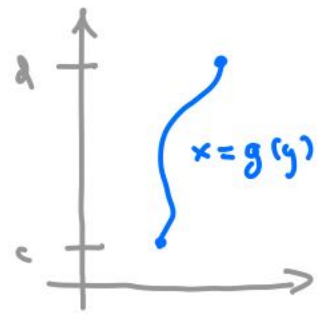
$$= \sqrt{17} + \frac{1}{4}\ln(\sqrt{17}+4).$$

$$\approx 4.8356.$$

A general formula for the arclength of $y=f(x)$ from $x=a$ to b emerges from this first example:

$$\boxed{L = \int_a^b \sqrt{1+(f'(x))^2}\,dx \quad \text{or} \quad \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}$$

One can also "flip this around" if x is given as a function of y:



$$L = \int_c^d \sqrt{1+(g'(y))^2}\,dy$$
$$= \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy.$$

Notice that both formulas suggest an integral of

$$\text{"}\sqrt{(dx)^2+(dy)^2}\text{"},$$

which is sometimes denoted "ds" for "differential of arclength". (The "s" comes from Latin: "Spatium".)

The second thing you see from the example is that we get a complicated integral from a "nice" function. Sometimes "nastier" functions give nicer answers.

TRY: Find the length of $x = \underbrace{y^2/4 - \frac{1}{2}\ln y}_{g(y)}$ from $y = 1$ to $y = \sqrt{e}$.

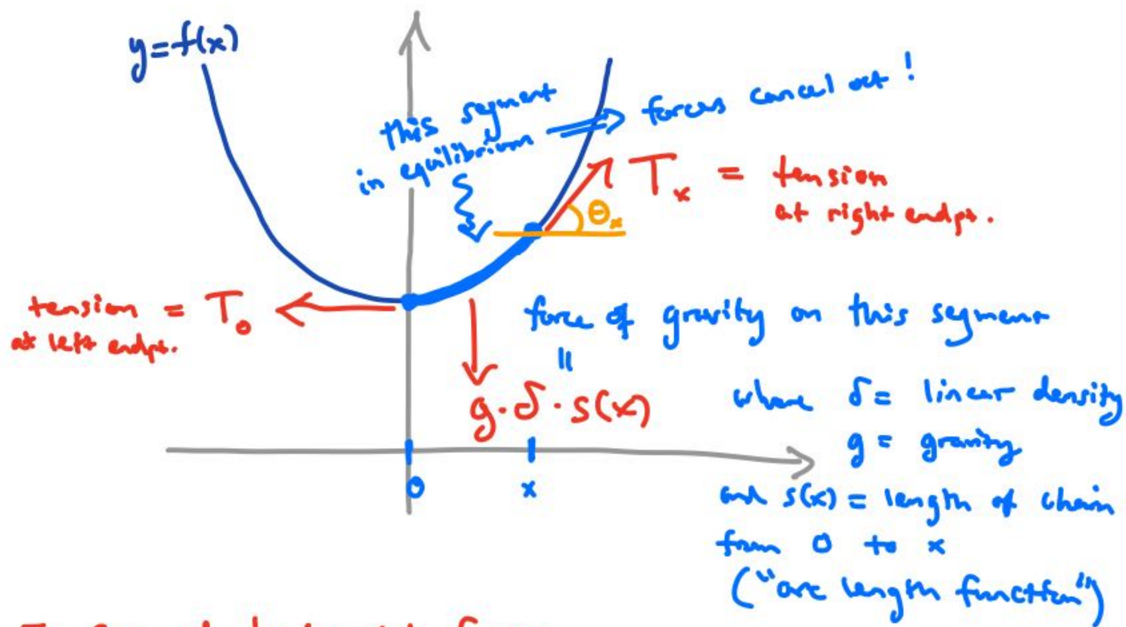
Ans: $$L = \int_1^{\sqrt{e}} \sqrt{1+(g'(y))^2}\,dy = \int_1^{\sqrt{e}} \sqrt{1+\left(y/2 - 1/2y\right)^2}\,dy$$
$$= \int_1^{\sqrt{e}} \sqrt{\frac{y^2}{4} + \underbrace{1 - \frac{1}{2}}_{\frac{1}{2}} + \frac{1}{4y^2}}\,dy = \int_1^{\sqrt{e}} \sqrt{\left(y/2 + \frac{1}{2y}\right)^2}\,dy$$

$$= \int_1^{\sqrt{e}} \left(\frac{y}{2} + \frac{1}{2y}\right) dy = \left[\frac{y^2}{4} + \frac{1}{2}\ln|y|\right]_1^{\sqrt{e}}$$

$$= \frac{e}{4} + \cancel{\frac{1}{2}\ln(e^{1/2})} - \cancel{\frac{1}{4}} - \cancel{\frac{1}{2}\ln 1}^{\,0} = \frac{e}{4}.$$

Ex 2/ Of what function is a freely hanging wire/chain (with only the two ends fixed) the graph?

You may have heard of a "catenary", and know that the Arch is a flattened Catenary, but what sort of curve is that?



- 0 = sum of horizontal forces
  $= T_x \cos\theta_x - T_0 \rightsquigarrow \boxed{T_x \cos\theta_x = T_0}$

- 0 = sum of vertical forces
  $= T_x \sin\theta_x - g\delta s(x) \rightsquigarrow \boxed{T_x \sin\theta_x = g\delta s(x)}$

Dividing the boxed equations gives $\tan\theta_x = \frac{g\delta}{T_0} s(x)$,

Call this "a"

and $\tan\theta_x$ is after all just $f'(x)$.

So we have

$$f'(x) = a \cdot \underbrace{s(x)}_{} = a \underbrace{\int_0^x \sqrt{1+(f'(t))^2}\, dt}$$

$\frac{d}{dx}$

$$f''(x) = a\sqrt{1+(f'(x))^2}$$

$$\text{or} \quad \boxed{(f''(x))^2 = a^2 + a^2 (f'(x))^2.} \qquad (*)$$

You may be familiar with the facts that

$$(\sinh(x))^2 + 1 = (\cosh(x))^2 \quad \text{and} \quad \begin{cases} \cosh'(x) = \sinh(x) \\ \sinh'(x) = \cosh(x). \end{cases}$$

These imply at once that

$$f(x) = \frac{1}{a}\cosh(ax)$$

solves $(*)$, and is (up to $+C$) the unique solution.

(The Gateway Arch is about $\frac{2}{3} \times$ such an f, flipped over.) //

TRY Find the "arclength function" $s(x)$ for the curve $y = \frac{2}{3}x^{3/2}$, i.e. the length of this curve from $(0,0)$ to $(x, \frac{2}{3}x^{3/2})$.

Ans.: $s(x) = \int_0^x \sqrt{1+(f'(t))^2}\, dt = \int_0^x \sqrt{1+(t^{1/2})^2}\, dt$

$$= \int_0^x \sqrt{1+t}\, dt = \left[\frac{2}{3}(1+t)^{3/2}\right]_0^x = \frac{2}{3}\left\{(1+x)^{3/2} - 1\right\}.$$