Lecture 25: Surface Area

We will only be dealing with surfaces of revolution (the more general case being the domain of Calculus III).

Ex 1/ A fraction of a cone is the surface between 2 planes perpendicular to the axis of the cone.

In radians, we have that

\[ \theta = \frac{2\pi r_2}{L_2} = \frac{2\pi r_1}{L_1}. \]

So the area of the large sector is

\[ \frac{\theta}{2\pi} \times (\text{area of circle of radius } L), \]

i.e. \[ \frac{r_2^2}{L_2} \times \pi L_2^2 = \pi r_2 L_2. \]

The area of the small sector is \[ \frac{r_1^2}{L_1} \times \pi L_1^2 = \pi r_1 L_1. \]

Here \( A = \pi r_2 L - \pi r_1 L_1 = \pi \left\{ r_2 L + r_2 L_1 - r_1 L_1 \right\} \)

\[ \therefore \text{use } \frac{r_1}{r_2} = \frac{L_1}{L_2}. \]
We write this result

\[ A \text{ (frustum)} = 2\pi \left( \text{average radius} \right) \left( \text{slant height} \right) \]

Now, revolve the curve \([y = f(x), \ a < x < b]\) about the \(x\)-axis and chop the surface area into bits:

\[ y = f(x) \]

These bits are approximately mini-frustums with slant height \(l = \Delta s\) and \(r_1 = f(x_{i-1}), \ r_2 = f(x_i)\). So the \(i\)-th bit has area

\[ (\Delta A)_i \approx 2\pi \frac{f(x_{i-1}) + f(x_i)}{2} \Delta s = 2\pi f(x_i^*) (\Delta s), \]

for some \(x_i^* \in (x_{i-1}, x_i)\) and where

\[ (\Delta s)_i \approx \sqrt{1 + (f'(x_i^*))^2} \Delta x \]

as in the last lecture. Adding the \((\Delta A)_i\)'s approximates

\[ A = \sum_{i=1}^{n} 2\pi f(x_i^*) \sqrt{1 + (f'(x_i^*))^2} \Delta x \]

by a Riemann sum, so that taking \(n \to \infty\) yields
\[ A = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2} \, dx \quad \left( = \int_{a}^{b} 2\pi r \, ds \right) \]

**Ex 2** Find the area of the surface of revolution generated by revolving the curve \( y = \sqrt{x} \), \( 0 \leq x \leq 4 \), about the \( x \)-axis.

Write \( f(x) = \sqrt{x} \), \( f'(x) = \frac{1}{2\sqrt{x}} \), so

\[
A = 2\pi \int_{0}^{4} \sqrt{x} \sqrt{1 + \frac{1}{4x}} \, dx = \pi \int_{0}^{4} \sqrt{4x + 1} \, dx
\]

\[
= \frac{\pi}{4} \left[ \frac{4}{3} u^{3/2} \right]_{1}^{17} = \frac{\pi}{4} \frac{4^{3/2}}{3^{3/2}} \left( 17^{3/2} - 1 \right).
\]

\[
\frac{4}{4} \, du = dx
\]

\[
u = 4x + 1
\]

\[
u = 4x + 1
\]

**Ex 3** Find the surface area of Gabriel's horn (surface of revolution of \( y = \frac{1}{x} \) about \( x \)-axis, with \( x > 1 \)).

Write \( f(x) = \frac{1}{x} \), \( f'(x) = -\frac{1}{x^2} \), so

\[
A = 2\pi \int_{1}^{\infty} \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} \, dx
\]

\[
\int_{1}^{\infty} \frac{1}{x} \, dx
\]

diverges!

So the surface area is infinite!
Try: Find the area of the surface obtained by

revolving \( x = \sqrt{4-y^2}, \quad -1 \leq y \leq 1 \), about the y-axis.

\[ A = \int_{-1}^{1} 2\pi \sqrt{4-y^2} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dy \]

\[ = \int_{-1}^{1} 2\pi \sqrt{4-y^2} \left\{ 1 + \left( \frac{-y}{\sqrt{4-y^2}} \right)^2 \right\} \, dy \]

\[ = \int_{-1}^{1} 2\pi \sqrt{4-y^2} \left( 1 + \frac{y^2}{4-y^2} \right) \, dy = \int_{-1}^{1} 2\pi \sqrt{4-y^2} \, dy \]

\[ = \int_{-1}^{1} 2\pi \sqrt{4-y^2} \, dy = 4\pi \left[ y - \frac{y^3}{6} \right]_{-1}^{1} = 4\pi - 2 = 8\pi. \]

Ex 4: Surface area of a sphere of radius \( a \):

Take \( x = g(y) = \sqrt{a^2-y^2}, \quad g'(y) = \frac{-y}{\sqrt{a^2-y^2}} \)

\[ A = \int_{-a}^{a} 2\pi \sqrt{a^2-y^2} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dy = \int_{-a}^{a} 2\pi \sqrt{a^2-y^2+y^2} \, dy = \int_{-a}^{a} 2\pi \sqrt{a^2} \, dy = 4\pi a^2. \]
Ex 5/ \[ \text{Area of surface obtained by revolving } x = \sqrt{a^2 - y^2}, \] 
\[ 0 \leq y \leq a, \text{ about the line } x = a: \]

\[ A = 2\pi \int_0^a \left( a - \sqrt{a^2 - y^2} \right) \sqrt{1 + \left( \frac{y}{\sqrt{a^2 - y^2}} \right)^2} \, dy \]

\[ = 2\pi \int_0^a \left( a - \sqrt{a^2 - y^2} \right) \frac{a}{\sqrt{a^2 - y^2}} \, dy \]

\[ = 2\pi a^2 \int_0^a \frac{dy}{\sqrt{a^2 - y^2}} - 2\pi a \int_0^a dy \]

\[ = 2\pi a^2 \left( \text{arcsin} \left( \frac{y}{a} \right) \right) \bigg|_0^a - 2\pi a^2 \]

\[ = 2\pi a^2 \left( \text{arcsin} \left( 1 \right) - \text{arcsin} \left( 0 \right) \right) - 2\pi a^2 \]

\[ = 2\pi a^2 \left( \frac{\pi}{2} - 1 \right). \]