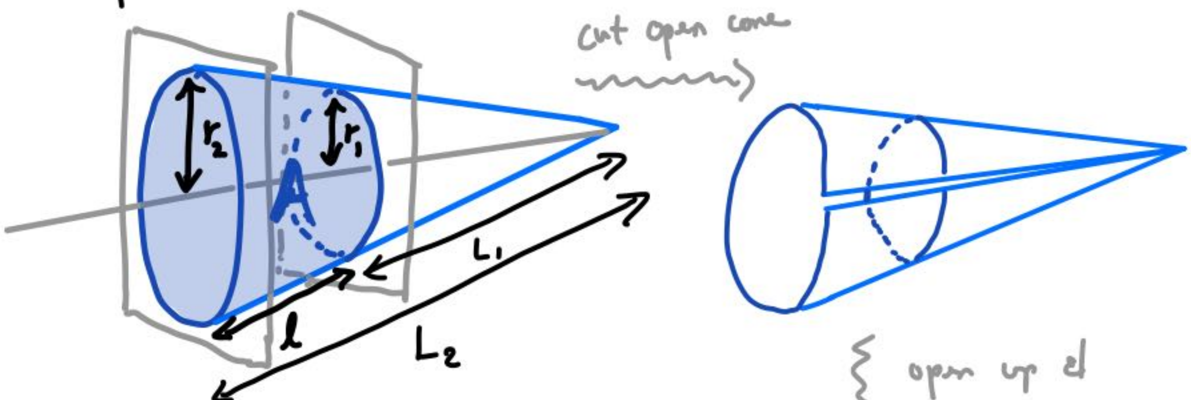
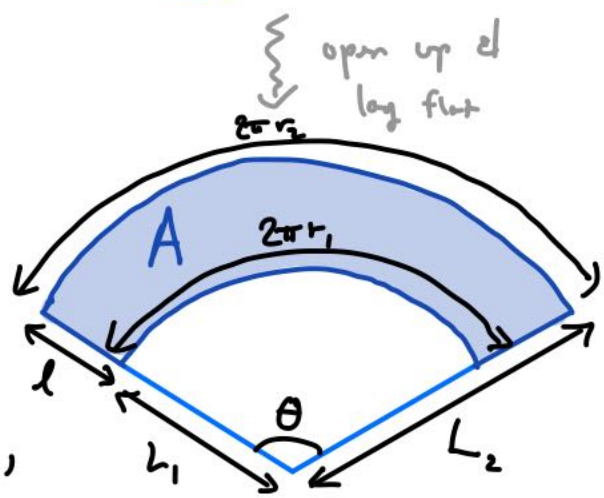
Lecture 25: Surface Area

We will only be dealing with surfaces of revolution (the more general case being the domain of Calculus III).

Ex 1/ A <u>frustum of a cone</u> is the surface between 2 planes perpendicular to the axis of the cone.





In radians, we have that

$$\theta = \frac{2\pi r_2}{L_2} = \frac{2\pi r_1}{L_1}.$$

So the area of the large sector is $\frac{\theta}{2\pi} \times$ (area of circle of radius L),

i.e. $\frac{r_2}{L_2} \times \pi L_2^2 = \pi r_2 L_2$.

The area of the small sector is $\frac{r_1}{L_1} \times \pi L_1^2 = \pi r_1 L_1$.

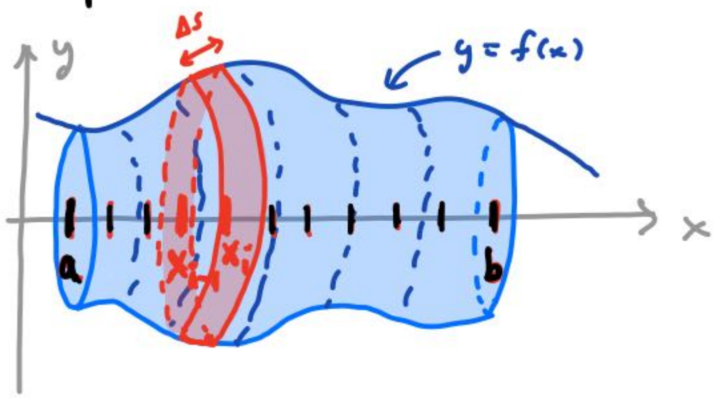
Hence $A = \pi r_2 L - \pi r_1 L_1 = \pi \{ r_2 \ell + r_2 L_1 - r_1 L_1 \}$ use $\frac{r_1}{r_2 - r_1} = \frac{L_1}{\ell}$

$$= \pi \left\{ r_2 \ell + \cancel{(r_2 - r_1)} \frac{r_1 \ell}{\cancel{r_2 - r_1}} \right\} = \pi (r_1 + r_2) \ell = 2\pi \cdot \frac{r_1 + r_2}{2} \cdot \ell. \quad //$$

We write this result

$$\boxed{A(\text{frustum}) = 2\pi(\text{average radius})(\text{slant height})}$$

Now revolve the curve $\{y = f(x),\ a \le x \le b\}$ about the x-axis and chop the surface area into bits:



these bits are approximately mini-frustums with slant height $l = \Delta s$ and $r_1 = f(x_{i-1})$, $r_2 = f(x_i)$. So the i^{th} bit has area

$$(\Delta A)_i \approx 2\pi \frac{f(x_{i-1}) + f(x_i)}{2} \Delta s = 2\pi f(x_i^*)(\Delta s)_i$$

(use intermediate value theorem)

for some $x_i^* \in (x_{i-1}, x_i)$ and where

$$(\Delta s)_i \approx \sqrt{1 + (f'(x_i^*))^2}\, \Delta x$$

as in the last lecture. Adding the $(\Delta A)_i$ up approximates

$$A \approx \sum_{i=1}^{n} 2\pi f(x_i^*) \sqrt{1 + (f'(x_i^*))^2}\, \Delta x$$

by a Riemann sum, so that taking $n \to \infty$ yields

$$A = 2\pi \int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx \quad \left(= \text{"}\int_*^{**} 2\pi r\,ds\text{"} \right)$$

Ex 2/ Find the area of the surface of revolution generated by revolving the curve $y=\sqrt{x}$, $0 \le x \le 4$, about the x-axis.

Write $f(x)=\sqrt{x}$, $f'(x)=\frac{1}{2\sqrt{x}}$, so

$$A = 2\pi\int_0^4 \sqrt{x}\sqrt{1+\frac{1}{4x}}\,dx = \pi\int_0^4 \sqrt{4x+1}\,dx$$

$$= \frac{\pi}{4}\int_1^{17} u^{1/2}\,du = \frac{\pi}{\cancel{4}_{2}}\,\frac{u^{3/2}}{3/\cancel{2}}\Bigg|_1^{17} = \frac{\pi}{6}\left(17^{3/2}-1\right).$$

$u = 4x+1$

$\frac{1}{4}du = dx$

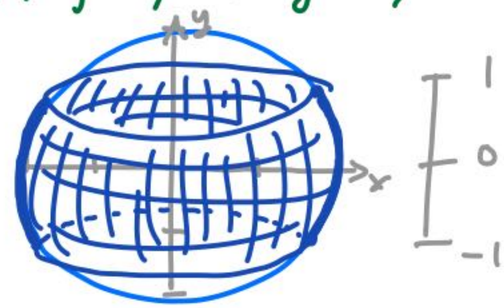
Ex 3/ Find the surface area of Gabriel's horn (surface of revolution of $y=\frac{1}{x}$ about x-axis, with $x \ge 1$).

Write $f(x)=\frac{1}{x}$, $f'(x)=\frac{-1}{x^2}$, so

$$A = 2\pi\int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx \ge \underbrace{2\pi\int_1^\infty \frac{1}{x}\,dx}_{\text{diverges!}}$$

So the surface area is infinite!

TRY: Find the area of the surface obtained by revolving $x = \sqrt{4-y^2}$, $-1 \le y \le 1$, about the y-axis:



Ans: Reverse the variables: "$\int 2\pi r\, ds$" becomes

$$(A =) \int_{-1}^{1} 2\pi \sqrt{4-y^2} \sqrt{1 + \left(\frac{d}{dy}\sqrt{4-y^2}\right)^2}\, dy$$

$$= \int_{-1}^{1} 2\pi \sqrt{(4-y^2)\left\{1 + \left(\frac{-y}{\sqrt{4-y^2}}\right)^2\right\}}\, dy$$

$$= \int_{-1}^{1} 2\pi \sqrt{(4-y^2)\left(1 + \frac{y^2}{4-y^2}\right)}\, dy = \int_{-1}^{1} 2\pi \sqrt{4-y^2+y^2}\, dy$$

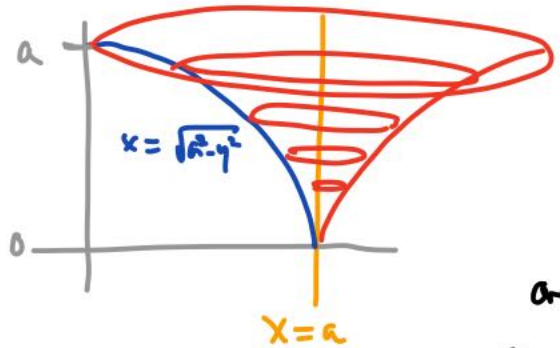
$$= \int_{-1}^{1} 2\pi \sqrt{4}\, dy = 4\pi \int_{-1}^{1} dy = 4\pi \cdot 2 = 8\pi.$$

Ex 4/ Surface area of a sphere of radius a:

take $x = g(y) = \sqrt{a^2-y^2}$, $g'(y) = \dfrac{-y}{\sqrt{a^2-y^2}}$ $\Longrightarrow$

$$A = \int_{-a}^{a} 2\pi \sqrt{a^2-y^2} \sqrt{1 + \frac{y^2}{a^2-y^2}}\, dy = \int_{-a}^{a} 2\pi \sqrt{a^2-y^2+y^2}\, dy = \int_{-a}^{a} 2\pi a\, dy = 4\pi a^2. \quad //$$

Ex 5 / Area of surface obtained by revolving $x = \sqrt{a^2-y^2}$, $0 \le y \le a$, about the line $x = a$:



Once again, we must add up areas of frustums, or use the general formula " $\int 2\pi r\, ds$ "

$$A = 2\pi \int_0^a \underbrace{\left(a - \sqrt{a^2-y^2}\right)}_{r(y)} \underbrace{\sqrt{1 + \left(\frac{-y}{\sqrt{a^2-y^2}}\right)^2}\, dy}_{ds}$$

$$= 2\pi \int_0^a \left(a - \sqrt{a^2-y^2}\right) \frac{a}{\sqrt{a^2-y^2}}\, dy$$

$$= 2\pi a^2 \int_0^a \frac{dy}{\sqrt{a^2-y^2}} - 2\pi a \int_0^a dy$$

$$= 2\pi a^2 \arcsin\left(y/a\right)\Big|_0^a - 2\pi a^2$$

$$= 2\pi a^2 \left(\underbrace{\arcsin(1)}_{\pi/2} - \underbrace{\arcsin(0)}_{0}\right) - 2\pi a^2$$

$$= 2\pi a^2 \left(\frac{\pi}{2} - 1\right).$$

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