Lecture 27 : Sequences

A sequence for us will be an infinite ordered list of numbers, like

$$\underset{a_1}{-1}, \underset{a_2}{2}, \underset{a_3}{-3}, \underset{a_4}{4}, \underset{a_5}{-5}, \underset{a_6}{6}, \ldots \qquad \{a_n\}$$

$$\underset{b_1}{1/3}, \underset{b_2}{1/4}, \underset{b_3}{1/5}, \underset{b_4}{1/6}, \underset{b_5}{1/7}, \ldots \qquad \{b_n\}$$

$$\underset{c_1}{3}, \underset{c_2}{12}, \underset{c_3}{27}, \underset{c_4}{48}, \ldots \qquad \{c_n\}$$

The general question we want to ask is: <u>where are these going</u>? By this I mean both being able to predict

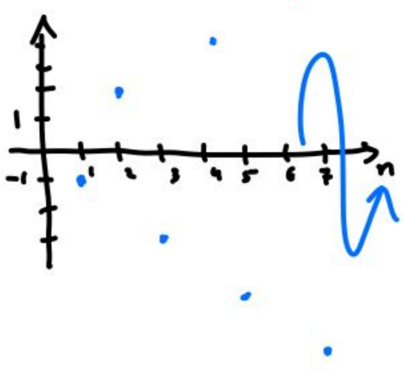
(a) their value at any given term (e.g., $1000^{\underline{th}}$)

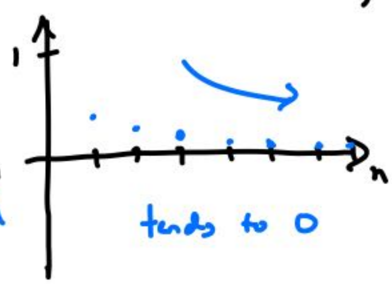
(b) to what value they tend.

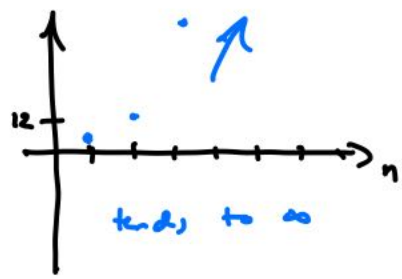
To this end it helps to write a formula for the $n^{\underline{th}}$ term:

$$a_n = (-1)^n \cdot n \quad , \quad b_n = 1/(n+2) \quad , \quad c_n = 3 \cdot n^2$$

$$\Rightarrow a_{1000} = 1000 \quad , \quad b_{1000} = 1/1002 \quad , \quad c_{1000} = 3{,}000{,}000$$







A bit more precisely, by (b) we mean "what is $\lim_{n\to\infty} a_n$"?

We write $\boxed{\lim_{n\to\infty} a_n = L}$ or $\boxed{a_n \to L}$, and say that a_n converges to L if we can ensure that a_n is arbitrarily close to L by going out far enough in the sequence. This is preferable to the less precise "a_n gets closer and closer to L" (yes, but it could still limit to $L+1$) or the absurd "when $n =$ godzillion, $a_n = L$" (∞ is not a number). If "$\lim_{n\to\infty} a_n = L$" holds for no finite L, as for both $\{a_n\}$ and $\{c_n\}$ above, we say the sequence diverges.

Remark: Of course, there is a difference between $\{a_n\}$ and $\{c_n\}$ on the first page. We say that $\boxed{\lim_{n\to\infty} c_n = \infty}$ since we can ensure that c_n is arbitrarily large by going out far enough in the sequence.

There are a bunch of tools for computing these limits:

- limits are "linear": if $\{a_n\}$ & $\{b_n\}$ converge, then
$$\lim_{n\to\infty} (C\cdot a_n + D\cdot b_n) = C\cdot\lim_{n\to\infty} a_n + D\cdot\lim_{n\to\infty} b_n .$$
- products & quotients: if $\{a_n\}$ & $\{b_n\}$ converge, then
$$\lim_{n\to\infty} (a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$$
and $$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$$ provided $\lim_{n\to\infty} b_n \neq 0$.
- powers: $\lim_{n\to\infty} n^\alpha = \begin{cases} \infty & \text{if } \alpha > 0 \\ 0 & \text{if } \alpha < 0 \end{cases}$, $\lim_{n\to\infty} R^n = \begin{cases} \infty & \text{if } R > 1 \\ 0 & \text{if } 0 < R < 1. \end{cases}$
- squeeze: if $a_n \le b_n \le c_n$ and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$, then $\lim_{n\to\infty} b_n = L$.

- Continuous functions: if $\lim_{n\to\infty} a_n = L$ and $f(x)$ is continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$.

You can already solve some problems with just these rules:

Ex 1/ $$\lim_{n\to\infty} \frac{1}{1+n^3} = \lim_{n\to\infty} \frac{1/n^3}{1/n^3+1} = \frac{\lim_{n\to\infty} n^{-3}}{\lim_{n\to\infty}(n^{-3}+1)} = \frac{0}{0+1} = 0 \quad //$$

Ex 2/ $$\lim_{n\to\infty} \frac{\sqrt{e^{2n}-e^{-2n}}}{1+e^n} = \lim_{n\to\infty} \frac{\sqrt{1-e^{-4n}}}{e^{-n}+1} = \frac{\sqrt{\lim_{n\to\infty}\left(1-(e^{-4})^n\right)}}{\lim_{n\to\infty}\left((e^{-1})^n+1\right)}$$

($\sqrt{\ }$ is continuous at 1)

$$= \frac{\sqrt{1-0}}{0+1} = 1. \quad //$$

But a more powerful tool for your arsenal is this:

- Suppose that $a_n = f(n)$ for some function f, and $\lim_{x\to\infty} f(x) = L$. Then also $\lim_{n\to\infty} a_n = L$.

You're going to use this in conjunction with L'Hôpital's rule, of which I now offer a brief review.

Recall that if $\lim_{x\to\infty} f(x)$, $\lim_{x\to\infty} g(x)$ are both 0 or both ∞, we have

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} \overset{\text{"L'H"}}{=} \lim_{x\to\infty} \frac{f'(x)}{g'(x)} .$$

Other "indeterminate forms" than $\frac{0}{0}$ and $\frac{\infty}{\infty}$ may be converted into one of these forms:

- "$0 \cdot \infty$": rewrite $\underset{\to 0}{f(x)}\,\underset{\to\infty}{g(x)}$ as $\dfrac{f(x) \to 0}{(1/g(x)) \to 0}$

- "$\infty - \infty$": rewrite $f(x) - g(x)$ as $\left(\frac{f(x)}{g(x)} - 1\right) g(x) = \frac{\left(\frac{f(x)}{g(x)} - 1\right)}{1/g(x)}$

  ... with the exception of $\ln F(x) - \ln G(x)$, which is simply $\ln(F(x)/G(x))$.

- $0^0, \infty^0, 1^\infty$: assume $\lim_{x\to\infty} f(x)^{g(x)} = L$, take ln of both sides to have $\lim_{x\to\infty} g(x)\cdot \ln(f(x)) = \ln(L)$. The left-hand limit is now of the form $0\cdot\infty$.

Ex 1/ $\lim_{n\to\infty} \frac{\ln(n)}{\sqrt{n}} = \lim_{x\to\infty} \frac{\ln(x)}{x^{1/2}} \overset{L'H}{=} \lim_{x\to\infty} \frac{x^{-1}}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$ //

Ex 2/ $\lim_{n\to\infty} \frac{\sin(n)}{n} = \ldots 0$! Not a L'Hôpital problem:

from $-1 \le \sin n \le 1$, $-\frac{1}{n} \le \frac{\sin(n)}{n} \le \frac{1}{n}$ (squeeze theorem)

$$-\frac{1}{n} \to 0,\quad \frac{1}{n} \to 0 \quad\Rightarrow\quad \frac{\sin(n)}{n} \to 0$$ //

Ex 3/ $\lim_{n\to\infty} n \sin\left(\frac{\pi}{n}\right) = \lim_{x\to\infty} x \sin\left(\frac{\pi}{x}\right) = \lim_{x\to\infty} \frac{\sin\left(\frac{\pi}{x}\right)}{1/x}$

$\overset{L'H}{=} \lim_{x\to\infty} \frac{-\frac{\pi}{x^2}\cos\left(\frac{\pi}{x}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} \pi \cos\left(\frac{\pi}{x}\right) = \pi\cos(0)$

cos is continuous at 0

$= \pi$. //

Ex 4/ $\lim_{n\to\infty} \ln(2n^2+1) - \ln(n^2+n+2) = \lim_{n\to\infty} \ln\left(\frac{2n^2+1}{n^2+n+2}\right)$

$= \lim_{n\to\infty} \ln\left(\frac{2+n^{-2}}{1+n^{-1}+2n^{-2}}\right) = \ln\left(\frac{\lim_{n\to\infty}(2+n^{-2})}{\lim_{n\to\infty}(1+n^{-1}+2n^{-2})}\right)$

$= \ln\left(\frac{2}{1}\right) = \ln 2$ //

Ex 5/ $\lim_{n\to\infty}(1+\ln(n))^{1/n} = \lim_{x\to\infty}(1+\ln(x))^{1/x} =: L$

$$\ln L = \lim_{x\to\infty}\frac{\ln(1+\ln(x))}{x} \overset{L'H}{=} \lim_{x\to\infty}\frac{\frac{1}{1+\ln(x)}\cdot\frac{1}{x}}{1} = \frac{0\cdot 0}{1} = 0$$

$\Rightarrow L = e^0 = 1.$ //

TRY: $\lim_{n\to\infty}\sqrt{n+1}-\sqrt{n}$

Ans: $$\lim_{n\to\infty}\left(\frac{\sqrt{n+1}}{\sqrt{n}}-1\right)\sqrt{n} = \lim_{x\to\infty}\frac{\left(\sqrt{1+\frac{1}{x}}-1\right)}{1/\sqrt{x}} \overset{L'H}{=} \lim_{x\to\infty}\frac{\frac{1}{\cancel{2}\sqrt{1+\frac{1}{x}}}\cdot\frac{\cancel{-}1}{x^2}}{\frac{\cancel{-}1}{\cancel{2}x^{3/2}}}$$

$$= \lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{x}}}\cdot\frac{1}{\sqrt{x}} = \lim_{x\to\infty}\frac{1}{\sqrt{x+1}} = 0.$$

TRY: You are eating dinner at a circular table seating 1000, or maybe 10000. After dinner, you randomly change places for dessert. Estimate the probability that no one at the table sits next to the same person twice. (Assume the dependence of various diners' avoiding their former neighbors may be ignored.)

Ans: $$\lim_{n\to\infty}\left(\frac{n-2}{n}\right)^n = \lim_{x\to\infty}\left(1-\frac{2}{x}\right)^x =: L$$

$$\Rightarrow \ln(L) = \lim_{x\to\infty} x\cdot\ln\left(1-\frac{2}{x}\right) = \lim_{x\to\infty}\frac{\ln\left(1-\frac{2}{x}\right)}{1/x}$$

$$\overset{L'H}{=} \lim_{x\to\infty}\frac{\frac{1}{1-2/x}\cdot\frac{2}{x^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty}\frac{-2}{1-\frac{2}{x}} = -2 \Rightarrow L = e^{-2} \approx 0.135$$

$\Rightarrow \sim 13.5\%.$