Lecture 28: Fibonacci's rabbits and the Golden Ratio

<u>Recall</u>: a sequence is an infinite set of real numbers listed in order as $a_1, a_2, a_3, \ldots$ (or $\{a_n\}$ for short).

You can give a sequence by a straight-up formula $a_n = f(n)$, or by a <u>recurrence relation</u> expressing a_n in terms of preceding terms (a_{n-1}, a_{n-2}, etc.), like $a_n = F(a_{n-1})$ or as in this next

Ex 1/ A particular breed of rabbit is immortal, and every month each <u>pair</u> produces a new <u>pair</u> which becomes productive at 2 months. Fibonacci just picked up a newborn <u>pair</u> at the pet store. After quite a few months, at what rate will they be multiplying?

Let f_n = # of <u>pairs</u> at beginning of month n.
Then the # of <u>new</u> rabbit <u>pairs</u> one month later is the number of rabbit pairs <u>not</u> newborn now: f_{n-1}. So

$$(*) \qquad f_{n+1} = f_n + f_{n-1}$$

yields the Fibonacci sequence: $f_1 = 1$, $f_2 = 1$, $f_3 = 2$, $f_4 = 3$, $f_5 = 5$, $f_6 = 8$, $f_7 = 13$, etc. The growth <u>rate</u> is f_{n+1}/f_n. Let's suppose it converges to some limit L. Then dividing $(*)$ by f_n,

$$\frac{f_{n+1}}{f_n} = 1 + \frac{f_{n-1}}{f_n} = 1 + \left(\frac{f_n}{f_{n-1}}\right)^{-1}$$

and taking limits gives

$$L = 1 + L^{-1} \Rightarrow L^2 - L - 1 = 0$$

$$\Rightarrow L = \frac{1+\sqrt{5}}{2}, \text{ the golden ratio! } //$$

I'll come back to recursion problems later.

Limits: The precise definitions are

- $\lim_{n\to\infty} a_n = L \iff \begin{cases} \text{for any } \epsilon > 0, \text{ there exists an integer } N \text{ such that} \\ n \geq N \Rightarrow |a_n - L| < \epsilon. \end{cases}$

- $\lim_{n\to\infty} a_n = \infty \iff \begin{cases} \text{for any } M > 0, \text{ there exists an integer } N \text{ such that} \\ n \geq N \Rightarrow a_n > M. \end{cases}$

Ex 2 / Here's how you show $\lim_{n\to\infty} r^n = 0$ if $|r| < 1$. For small ϵ, we must produce N large enough that for $n \geq N$, $|r^n| < \epsilon$. But taking ln of both sides gives $n \ln|r| < \ln \epsilon$. Since $\ln|r|$ and $\ln \epsilon$ are both negative, this is the same as $n > \frac{\ln \epsilon}{\ln|r|}$ (>0). So if we take N to be any integer larger than $\frac{\ln \epsilon}{\ln|r|}$, then $n \geq N \Rightarrow n > \frac{\ln \epsilon}{\ln|r|} \Rightarrow n\ln|r| < \ln \epsilon$

$\underset{\exp}{\Rightarrow} |r|^n < \epsilon$ as desired. //

TRY: Which of the following converge? $a_n =$

$(-1)^n$, $\left(-\frac{1}{2}\right)^n$, 2^n, $\frac{n}{\ln n}$, $\frac{n!}{n^n}$?

recall that $n! = n(n-1)(n-2)\cdots 2\cdot 1$.

[Ans: $\left(-\frac{1}{2}\right)^n$ and $\frac{n!}{n^n}$.]

The first three were all r^n's, $\frac{n}{\ln n}$ you show diverges by L'Hôpital, but what about $\frac{n!}{n^n}$? This motivates the introduction of one more tool.

Definition:

- $\{a_n\}$ is increasing if $a_{n+1} > a_n$, decreasing if $a_{n+1} < a_n$, and monotonic if one of those is true.
- $\{a_n\}$ is bounded below if $a_n \geq B$, bounded above if $a_n \leq C$, and bounded if both are true.

THEOREM: A bounded monotonic sequence is convergent.
(More precisely: bounded above + increasing, or bounded below + decreasing, will do.)

[Why is it true? Say $\{a_n\}$ is bounded below + decreasing. Let B be the greatest lower bound, so that for any $\epsilon > 0$ there is some a_N with $a_N < B + \epsilon$. Since a_n is decreasing,
$n \geq N \Rightarrow a_n < a_N < B + \epsilon \Rightarrow 0 \leq a_n - B < \epsilon \Rightarrow |a_n - B| < \epsilon$.
So $\lim_{n\to\infty} a_n = B$. (NOTE: That greatest lower bounds exist is a deep property of the real numbers, called completeness.)]

TESTS FOR MONOTONICITY:

- DIFFERENCE TEST: $a_{n+1} - a_n > 0$ or < 0 for all n
  increasing, decreasing
- RATIO TEST: $a_n > 0$ and $\frac{a_{n+1}}{a_n} > 1$ or < 1 for all n

Ex 3/ Coming back to $a_n = \dfrac{n!}{n^n}$, we consider ratios

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} = \frac{\cancel{(n+1)}\,\cancel{n(n-1)\cdots 1}}{\cancel{n(n-1)\cdots 1}}\cdot\frac{n^n}{(n+1)^{\cancel{n+1}}} = \left(\frac{n}{n+1}\right)^n < 1.$$

So a_n is decreasing, and also bounded below (by 0), hence converges (by the Theorem).

But we're overcomplicating this: notice that

$$0 \le a_n = \underbrace{\frac{n\cdot(n-1)\cdot\ldots\cdot 2}{n\cdot n\cdot\ldots\cdot n}}_{<1}\,\frac{1}{n} < \frac{1}{n}$$

$$0 \xrightarrow{\text{squeeze}} 0 \xleftarrow{n\to\infty} \frac{1}{n}$$

Recurrence relations: $a_{n+1} = f(a_n)$

- Simplest type: $a_{n+1} = r\cdot a_n$. Immediately yields formula $a_n = a_0\cdot r^n$.
- Small populations: $a_{n+1} = f(a_n)$ where $f(x) = b\,x\left(1 - \frac{x}{C}\right)$ (b: growth rate, C: capacity)
- equilibrium is when $a_{n+1} = f(a_n)$.
  Find equilibria by solving $x^* = f(x^*)$. If a problem asks you to "find the limit assuming one exists", they are asking for this solution x^*.
- an equilibrium x^* is stable if starting close to x^* makes $a_n \to x^*$. It turns out that x^* is stable if $|f'(x^*)| < 1$, though I won't go into this.

Ex 4/ Define a sequence recursively by $a_1 = 2$,
$$a_{n+1} = 1/(3 - a_n).$$

We have $a_2 = \frac{1}{3-2} = 1$, $a_3 = \frac{1}{3-1} = \frac{1}{2}$, $a_4 = \frac{1}{3-\frac{1}{2}} = \frac{2}{5}, \ldots$ So it appears to be decreasing. To show this inductively, assume $a_n < a_{n-1}$; then $3 - a_n > 3 - a_{n-1}$, hence $a_{n+1} = \frac{1}{3-a_n} < \frac{1}{3-a_{n-1}} = a_n$. (Done.) Next, since it's decreasing, the sequence remains less than 3; so $a_{n-1} < 3$ $\Rightarrow 3 - a_{n-1} > 0 \Rightarrow a_n = \frac{1}{3-a_{n-1}} > 0$. By the Theorem, $\{a_n\}$ converges.

TRY: TO what ??

Ans: Since $a_{n+1} = f(a_n)$ with $f(x) = \frac{1}{3-x}$, write

$$L = \frac{1}{3-L} \Rightarrow 3L - L^2 = 1 \Rightarrow L^2 - 3L + 1 = 0$$

$$\Rightarrow L = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2} \underset{0<a_n<3}{\Longrightarrow} L = \frac{3-\sqrt{5}}{2}.$$

Let's finish off with one more example, which arises when you continuously compound interest (like banks do):

Ex 5/ $L = \lim_{n\to\infty} \left(1 + \frac{\alpha}{n}\right)^n$

$$\ln L = \lim_{n\to\infty} n \ln\left(1+\frac{\alpha}{n}\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{\alpha}{x}\right)}{1/x}$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{\alpha}{x}} \cdot \frac{-\alpha}{x^2}}{-1/x^2} = \lim_{x\to\infty} \frac{\alpha}{1+\frac{\alpha}{x}} = \alpha$$

$L = e^{\ln L} = e^{\alpha}$. (If α is very small, this is $\approx 1+\alpha$.)