Lecture 29 : Infinite Series

Today we are going to take all the numbers in an infinite sequence $\{a_n\}$ and add them together.

Definition 1: An infinite series is an expression of the form $a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$.

To make sense of anything "infinite" in Calculus, we invoke limits. But let's first look at the corresponding finite sums.

Definition 2: The n^{th} partial sum of a series is

$$S_n := a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

So

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$\vdots$$

forms a sequence $\{S_n\}$, the sequence of n^{th} partial sums. Notice that it starts to look more and more like the expression in Definition 1 as it continues on. This motivates the key

Definition 3: The sum of the series $\sum_{n=1}^{\infty} a_n$ is defined to be the limit $S = \lim_{n\to\infty} S_n$ of the sequence of nth partial sums. We write

$$\boxed{\sum_{n=1}^{\infty} a_n = S}$$

and say the series converges to S. Unless, of course the sequence $\{S_n\}$ diverges, in which case we say the series $\sum_{n=1}^{\infty} a_n$ diverges as well.

Ex 1/ Some divergent series:

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + \cdots \quad (S_n = n \to \infty)$$

$$\sum_{n=1}^{\infty} (-1)^n = -1 + 1 + (-1) + 1 + (-1) + \cdots$$

$$\left(S_n = \begin{cases} -1 & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \right)$$

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \cdots$$

$$\left(S_n = \frac{n(n+1)}{2} \to \infty \right)$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots$$

("Harmonic series")

roughly: group as shown, get $> 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$ //

Ex 2/ Some convergent series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}$$

we won't show that in this course, but will show series converges

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \ln(2)$$

will see this later in the course

$$\sum_{n=1}^{\infty} \frac{3}{10^n} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots = 0.3 + 0.03 + 0.003 + \cdots$$

$$= 0.33333\ldots = \frac{1}{3}$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$$

to see this: $S_1 = \frac{1}{2}$

$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$

$\vdots$

$S_n = \dfrac{2^n - 1}{2^n} = 1 - 2^{-n}$

So $\lim_{n\to\infty} S_n = 1$. //

The last two series above are of a type called <u>geometric series</u>, which means one of the form

$$a + ar + ar^2 + ar^3 + \cdots\cdots. \quad (r = \text{"ratio"})$$

Here it is convenient to start our sequence at $n=0$, and write $a_n = a \cdot r^n$ (notice that $\frac{a_{n+1}}{a_n} = r$), so the series is $\sum_{n=0}^{\infty} a r^n$ with n^{th} partial sum

(1) $$S_n = \sum_{i=0}^{n} a r^i = a + ar + \cdots + ar^n .$$

Notice that

(2) $$r S_n = \sum_{i=0}^{n} a r^{i+1} = \quad ar + \cdots + ar^n + ar^{n+1}.$$

Subtracting (1) − (2) gives

$$(1-r) S_n = a - a r^{n+1} = a(1 - r^{n+1})$$

so that (if $r \neq 1$) ← (if $r = 1$, $S_n = na$ and the series diverges)

$$S_n = \frac{a(1-r^{n+1})}{1-r}.$$

This converges if and only if $|r| < 1$, as we know from our study of sequences. So we conclude:

- If $|r| \geq 1$, $\sum_{n=0}^{\infty} a r^n$ diverges.
- If $|r| < 1$, $$\sum_{n=0}^{\infty} a r^n = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{a(1-r^{n+1})}{1-r} = \frac{a}{1-r}.$$

Ex 3/ Examples of geometric series:

$$\frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots = \sum_{n=0}^{\infty} \frac{3}{10}\left(\frac{1}{10}\right)^n = \frac{3/10}{1 - 1/10} = \frac{3/10}{9/10} = \frac{1}{3}$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^n = \frac{1/2}{1 - 1/2} = 1$$

$\boxed{\text{TRY}: \ 6 - 4 + \frac{8}{3} - \frac{16}{9} + \frac{32}{27} - \dots}$

[Ans.: $\sum_{n=0}^{\infty} 6\left(-\frac{2}{3}\right)^n = \frac{6}{1+2/3} = \frac{6}{5/3} = \frac{18}{5} = 3.6$.]

We could even try functions: $1+x+x^2+x^3+\dots = \frac{1}{1-x}$. //
(works for $|x|<1$)

Actually, any time you have a repeating decimal you can use geometric series to represent the number as a fraction:

Ex 4/ $1.167167167\ldots = 1 + \frac{167}{1000} + \frac{167}{(1000)^2} + \frac{167}{(1000)^3} + \cdots$

$$= 1 + \sum_{n=0}^{\infty} \frac{167}{1000}\left(\frac{1}{1000}\right)^n = 1 + \frac{167/1000}{1 - 1/1000} = 1 + \frac{167}{999}$$

$$= 1166/999.$$

//

Why might you care about this? The book has an example about drug concentrations in the bloodstream: the idea is that perhaps each dose raises the concentration by α, whereas the time between doses dilutes the concentration, multiplying it by β (e.g. 0.3 for 30%). So after initial dose, the concentration $C_0 = \alpha$; after the next, $C_1 = \alpha + \beta C_0 = \alpha + \beta\alpha$; after another, $C_2 = \alpha + \beta C_1 = \alpha + \beta\alpha + \beta^2\alpha$. Eventually we reach $C_n = \alpha + \beta\alpha + \cdots + \beta^n\alpha$, and by now it's clear these are the partial sums of $\sum_{i=0}^{\infty} \alpha\beta^i$. Thus the concentration, if this regime is maintained forever, limits to $\frac{\alpha}{1-\beta}$.

Geometric series can come in fancy guises:

Ex 5/ $\displaystyle\sum_{n=1}^{\infty} \frac{3^n - 2^{n+2}}{5^{n+1}} = ?$ Be careful!!

The series starts at $n=1$, among other things...

$$\| $$

$$\sum_{n=1}^{\infty} \frac{3^n}{5^{n+1}} - \sum_{n=1}^{\infty} \frac{2^{n+2}}{5^{n+1}} = \sum_{m=0}^{\infty} \frac{3}{25}\left(\frac{3}{5}\right)^m - \sum_{m=0}^{\infty} \frac{8}{25}\left(\frac{2}{5}\right)^m$$

reindex: first series has $a = 3/5^2$, $r = \frac{3}{5}$
second has $a = 8/5^2$, $r = \frac{2}{5}$

$$= \frac{3/25}{1-3/5} - \frac{8/25}{1-2/5} = \frac{\cancel{5}}{2}\cdot\frac{3}{\cancel{25}_{5}} - \frac{\cancel{5}}{3}\cdot\frac{8}{\cancel{25}_{5}}$$

$$= \frac{3}{10} - \frac{8}{15} = \frac{-7}{30}. \quad //$$

$$\boxed{\text{TRY: } \sum_{n=1}^{\infty} \left(\left(\frac{1}{6}\right)^n - \left(\frac{1}{6}\right)^{n+1}\right) = ?}$$

We could break this into 2 pieces... but there's a better way!

Ans: $\left(\frac{1}{6} - \cancel{\left(\frac{1}{6}\right)^2}\right) + \left(\cancel{\left(\frac{1}{6}\right)^2} - \cancel{\left(\frac{1}{6}\right)^3}\right) + \left(\cancel{\left(\frac{1}{6}\right)^3} - \cancel{\left(\frac{1}{6}\right)^4}\right) + \cdots$

"telescoping sum" — more on this on Monday

$= \frac{1}{6}.$