Lecture 29: Infinite Series

Today we are going to take all the numbers in an infinite sequence \( \{a_n\} \) and add them together.

**Definition 1:** An infinite series is an expression of the form
\[
\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots
\]

To make sense of anything "infinite" in calculus, we invoke limits. But let's first look at the corresponding finite sums.

**Definition 2:** The \( n \)th partial sum of a series is
\[
S_n = a_1 + a_2 + \ldots + a_n = \sum_{i=1}^{n} a_i
\]

So
\[
S_1 = a_1 \\
S_2 = a_1 + a_2 \\
S_3 = a_1 + a_2 + a_3 \\
\vdots
\]

forms a sequence \( \{S_n\} \), the sequence of \( n \)th partial sums. Notice that it starts to look more
and more like the expression in Definition 1 as it continues on. This motivates the key.

**Definition 3:** The sum of the series $\sum_{n=1}^{\infty} a_n$ is defined to be the limit $S = \lim_{n \to \infty} S_n$ of the sequence of partial sums. We write

$$\sum_{n=1}^{\infty} a_n = S$$

and say the series converges to $S$. Unless of course the sequence $\{S_n\}$ diverges, in which case we say the series $\sum_{n=1}^{\infty} a_n$ diverges as well.

**Ex. 1** / Some divergent series:

\[ \sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + \cdots \quad (S_n = n \to \infty) \]

\[ \sum_{n=1}^{\infty} (-1)^n = -1 + 1 + (-1) + 1 + (-1) + \cdots \]

\( (S_n = \{ -1 \text{ if } n \text{ odd} \}) \)

\[ \sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \cdots \quad (S_n = \frac{n(n+1)}{2} \to \infty) \]

\[ \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots \]

(“Harmonic series”)

- roughly: group is shown, yet $\sum \frac{1}{n} > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \cdots$
Ex 2/ some convergent series:

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots = \frac{\pi^2}{6} \]

We won't show that in this course, but will show series converges.

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots = \ln(2) \]

will see this later in the course.

\[ \sum_{n=1}^{\infty} \frac{3}{10^n} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \ldots = 0.3 + 0.03 + 0.003 + \ldots = 0.3333\ldots = \frac{1}{3} \]

\[ \sum_{n=1}^{\infty} \left( \frac{1}{2} \right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = 1 \]

to see this:

\[ S_1 = \frac{1}{2} \]
\[ S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \]
\[ S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} \]
\[ \vdots \]
\[ S_n = \frac{2^n-1}{2^n} = 1 - 2^{-n} \]

so \( \lim_{n \to \infty} S_n = 1 \).

The last two series above are of a type called geometric series, which means one of the form

\[ a + ar + ar^2 + ar^3 + \ldots \]  \((r = "ratio")\)

here it is convenient to start our sequence at \( n = 0 \),
and write \( a_n = ar^n \) (note that \( \frac{a_n}{a_{n-1}} = r \)), so the series is \( \sum_{n=0}^{\infty} ar^n \) with \( n^{th} \) partial sum

1. \( S_n = \sum_{i=0}^{n} ar^i = a + ar + \ldots + ar^n \). Notice that

2. \( rS_n = \sum_{i=0}^{n} ar^{i+1} = ar + \ldots + ar^n + ar^{n+1} \).

Subtracting (1) - (2) gives

\[
(1-r)S_n = a - ar^{n+1} = a(1-r^{n+1})
\]

so that (if \( r \neq 1 \))

\[
S_n = \frac{a(1-r^{n+1})}{1-r}.
\]

This converges if and only if \( |r| < 1 \), as we know from our study of sequences. So we conclude:

- If \( |r| \geq 1 \), \( \sum_{n=0}^{\infty} ar^n \) diverges.
- If \( |r| < 1 \), \( \sum_{n=0}^{\infty} ar^n = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{a(1-r^{n+1})}{1-r} = \frac{a}{1-r} \).

Ex 3/ Examples of geometric series:

\[\frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \ldots = \sum_{n=0}^{\infty} \frac{3}{10} \left(\frac{1}{10}\right)^n = \frac{\frac{3}{10}}{1 - \frac{1}{10}} = \frac{3}{9} = \frac{1}{3}\]

\[\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{n=0}^{\infty} \frac{1}{2} \left(\frac{1}{2}\right)^n = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1\]
Try: \[ 6 \frac{1}{4} + \frac{8}{3} + \frac{16}{9} + \frac{32}{27} - \ldots \]

\[ \text{Ans.: } \frac{6}{1+2/3} = \frac{6}{5/3} = \frac{18}{5} = 3.6 \]

We could also try functions: \[ 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x} \quad (\text{valid for } |x|<1) \]

Actually, any time you have a repeating decimal you can use geometric series to represent the number as a fraction:

Ex 4/ \[ 1.167167167\ldots = 1 + \frac{167}{1000} + \frac{167}{(1000)^2} + \frac{167}{(1000)^3} + \ldots \]
\[ = 1 + \sum_{n=0}^{\infty} \frac{167}{1000} (\frac{1}{1000})^n = 1 + \frac{167}{1000} \left( \frac{1}{1-\frac{1}{1000}} \right) = 1 + \frac{167}{999} \]
\[ = 1.166\overline{6} \]

Why might you care about this? The book has an example about drug concentrations in the bloodstream: the idea is that perhaps each dose raises the concentration by \( \alpha \), whereas the time between doses dilutes the concentration, multiplying it by \( \beta \) (eg. 0.8 for 80%). So after initial dose, the concentration \( C_0 = \alpha \); after the next, \( C_1 = \alpha + \beta C_0 = \alpha + \beta \alpha \); after another, \( C_2 = \alpha + \beta C_1 = \alpha + \beta \alpha + \beta^2 \alpha \). Eventually we reach \( C_n = \alpha + \beta \alpha + \ldots + \beta^n \alpha \), and by now it's clear these are the partial sums of \( \sum_{i=0}^{\infty} \frac{\alpha \beta^i}{(1-\beta)} \). Thus, the concentration, if this regime is maintained forever, limits to \( \frac{\alpha}{1-\beta} \).
Geometric series can come in fancy guises:

Ex 5. \[
\sum_{n=1}^{\infty} \frac{3^n - 2^{n+2}}{5^n} = ? \quad \text{Be careful!!}
\]

The series starts at \(n=1\), among other things...

\[
\sum_{n=1}^{\infty} \frac{3^n}{5^n} - \sum_{n=1}^{\infty} \frac{2^{n+2}}{5^n} = \sum_{m=0}^{\infty} \frac{3}{25} \left(\frac{3}{5}\right)^m - \sum_{m=0}^{\infty} \frac{8}{25} \left(\frac{2}{5}\right)^m
\]

reindex: first series has \(a = \frac{3}{5}, r = \frac{3}{5}\)
second has \(a = \frac{8}{25}, r = \frac{2}{5}\)

\[
= -\frac{\frac{3}{25}}{1-\frac{3}{5}} - \frac{\frac{8}{25}}{1-\frac{2}{5}} = \frac{5}{25} - \frac{8}{25} \cdot \frac{5}{5}
\]

\[
= \frac{3}{10} - \frac{8}{15} = -\frac{7}{30}.
\]

TRY:

\[
\sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^n - \left(\frac{1}{6}\right)^{n+1} = ?
\]

We could break this into two pieces... but there's a better way!

\[
\sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^n - \left(\frac{1}{6}\right)^{n+1} = \frac{1}{6} - \left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right)^2 - \left(\frac{1}{6}\right)^3 + \left(\frac{1}{6}\right)^3 - \left(\frac{1}{6}\right)^4 + \ldots
\]

\[
= \frac{1}{6}.
\]

"telescoping sum" — man on this or

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