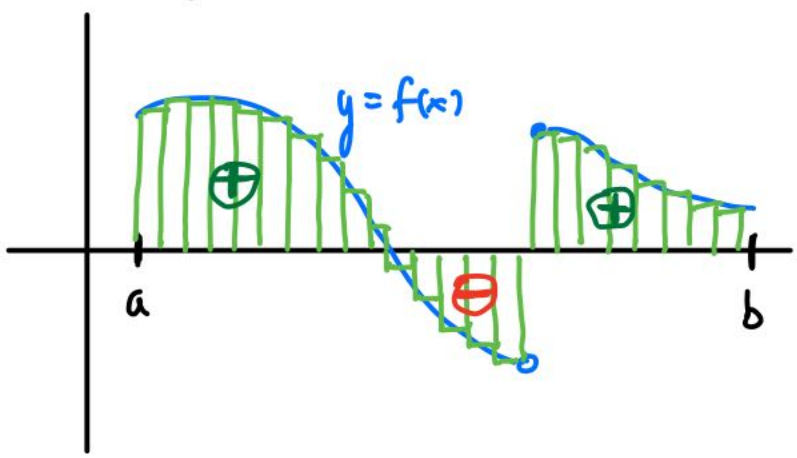
Lecture 3: The Definite Integral

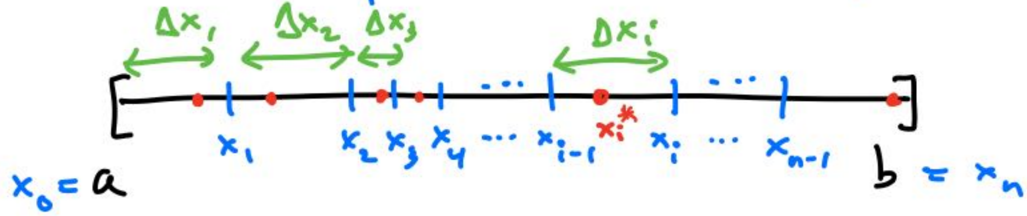
There is a general procedure for defining the "area under any curve" as a limit of sums (of areas of bars), like in Lecture 2. The area is counted as negative if the bar points down:



(The function need not even be continuous, as you can see; the widths of the bars need not be the same, either — we just require that the maximum width $\to 0$ as the # of bars $\to \infty$.)

To decide what the bars look like (and this really doesn't matter in the end), you take a partition of $[a,b]$:

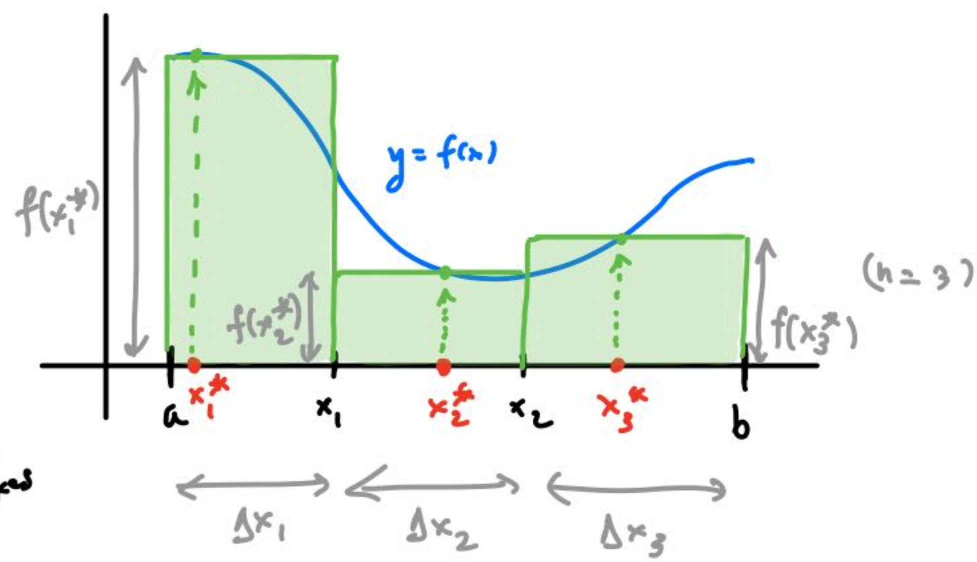
$$\underbrace{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b}_{\text{partition "}P\text{" (= choice of subdivision)}}$$



$$\Delta x_i = x_i - x_{i-1}$$

On each $[x_{i-1}, x_i]$, we also choose a sample point x_i^*.

We then take the bar over $[x_{i-1}, x_i]$ to be $f(x_i^*)$:



sum of areas of boxes

$$R_P := \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x_i$$ is then called a

Riemann sum for f corresponding to the partition P.

$\|P\|$:= Norm of P := the biggest Δx_i.

We say that f is integrable (on $[a,b]$) if

$\lim_{\|P\| \to 0} R_P$ exists (and is independent of the

e.g., by subdividing partitions

choices we made). In this case we write

big "$\int$" for "sum"

$$\boxed{\int_a^b \overbrace{f(x)\,dx}^{\text{“area of infinitesimal bar”}}}$$

"infinitesimal Δx"

"Definite integral of f on $[a,b]$"

for that limit.

All continuous functions are integrable. In fact, if f is

(a) bounded on $[a,b]$

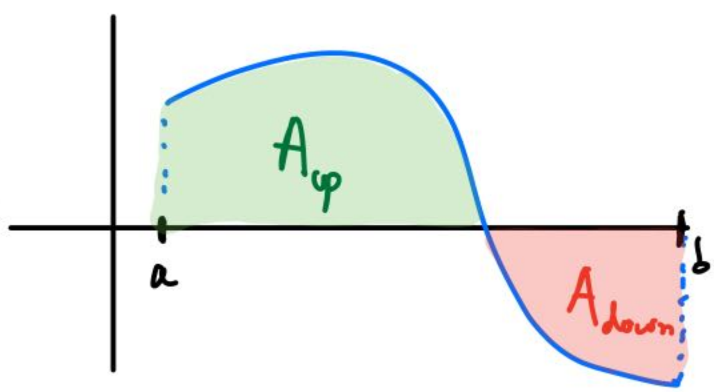
and

(b) discontinuous at no more than finitely many points,

it is integrable on $[a,b]$. Basically, this means to watch out at vertical asymptotes, but otherwise you're OK.

Intuitively, we have

$$\int_a^b f(x)\,dx = A_{up} - A_{down}:$$



To recapitulate: in Calculus I, you <u>defined</u> the slope of a graph (at a point) by a limit (of slopes of "secant lines" through 2 points). Here, we are <u>defining</u> the area under a graph as a limit (of sums of areas of bars). While "area" and "slope" might seem to be "physically self-explanatory", this is what it takes to make them into meaningful mathematical concepts, and even to compute them!

Ex/
$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} \sin(x_k^*)\, \Delta x_k = \int_0^{\pi} \sin(x)\,dx$$

$x_k - x_{k-1}$

sample point in $[x_{k-1}, x_k]$

partition of $[0,\pi]$ with n subintervals

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Ex/ Express $\int_2^6 \sqrt{x+1}\,dx$ as a limit of Riemann sums: taking P = partition of $[2,6]$, x_k^* = sample points,

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} \underbrace{\left(\sqrt{x_k^*+1}\right)}_{\text{height of box}} \underbrace{(\Delta x_k)}_{\text{width of box}}$$

//

For simplicity we'll always take all Δx_k's to be the same, $\Delta x = \dfrac{b-a}{n}$. Then $\|P\| \to 0$ just means $n \to \infty$, and

$$x_k = x_0 + k \cdot \Delta x = a + k\left(\frac{b-a}{n}\right).$$

(Usually we'll take $x_k^* = x_k,\ x_{k-1}$, or the midpoint.)

So the Riemann sum whose limit defines our integral is

$$R_n := \sum_{k=1}^{n} f(x_k^*)\,\Delta x \overset{x_k^* = x_k}{=} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k\left(\frac{b-a}{n}\right)\right)$$

$$\xrightarrow[n\to\infty]{} \int_a^b f(x)\,dx .$$

Ex/ Use the midpoint rule with $n=5$ to approximate $\displaystyle\int_0^2 \frac{x}{x+1}\,dx$: so $\Delta x = \dfrac{2}{5}$, $f(x) = \dfrac{x}{x+1}$,

$$x_k^* = \frac{x_k + x_{k-1}}{2} = \overset{0}{\cancel{a}} + \left(k - \tfrac{1}{2}\right)\frac{2}{5} = \frac{2k-1}{5}, \text{ and}$$

$$R_5 = \Delta x \sum_{k=1}^{5} f(x_k^*) = \frac{2}{5}\sum_{k=1}^{5} \frac{\frac{2k-1}{5}}{\frac{2k-1}{5}+1} = \frac{2}{5}\sum_{k=1}^{5}\frac{2k-1}{2k+4}$$

$$= \frac{2}{5}\left(\frac{1}{6} + \frac{3}{8} + \frac{5}{10} + \frac{7}{12} + \frac{9}{14}\right) \approx 0.907 \quad \ldots$$

not too bad an approximation to $2 - \ln(3) \approx 0.901$. //

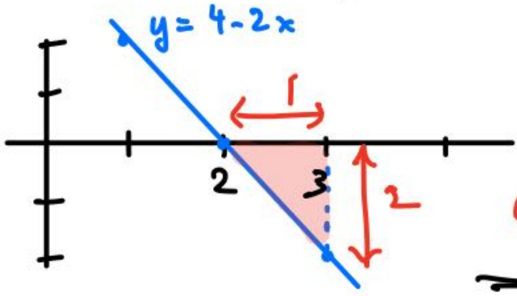
Ex/ Use the right-hand rule ("Theorem 4" in §5.2) to evaluate $\int_2^3 (4-2x)\,dx$.

$$R_n = \frac{1}{n}\sum_{i=1}^{n}\left(4 - 2\underbrace{x_i}_{2+\frac{i}{n}}\right) = \frac{1}{n}\sum_{i=1}^{n}\left(\cancel{4-2\cdot 2}-\frac{2i}{n}\right)$$

$$= -\frac{2}{n^2}\sum_{i=1}^{n} i = -\frac{\cancel{2}}{n^{\cancel{2}}}\,\frac{\cancel{n}(n+1)}{\cancel{2}} = -\frac{n+1}{n}$$

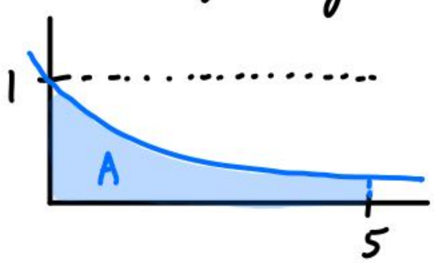
$$= -1 - \frac{1}{n} \xrightarrow[\lim_{n\to\infty}]{} \boxed{-1}. \quad //$$

Ex/ Use the area interpretation to evaluate $\int_2^3 (4-2x)\,dx$.



$\longrightarrow$ integral $= \boxed{-1}$. //

Ex/ What can you say about $\int_0^5 e^{-x}\,dx$ using areas?



It's between 0 and 5.

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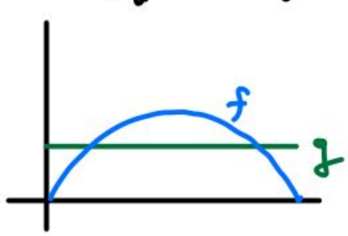
Properties of the definite integral

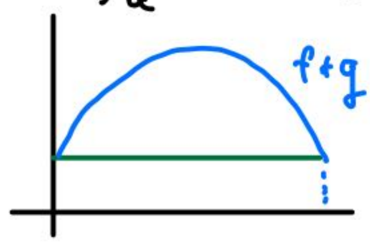
① Integration "over a point" is zero: $\int_a^a f(x)\,dx = 0$

② Integration "backwards" has negative Δx's, so is the $\ominus$ of "forwards" integral: $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

③ Integration is "linear": $\int_a^b C f(x)\,dx = C\int_a^b f(x)\,dx$,

and $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.





(clear in terms of areas)

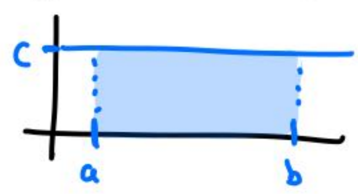
④ Integral of a positive function is the (positive) area under it: $f(x) \geq 0$ on $[a,b] \implies \int_a^b f(x)\,dx \geq 0$.

$\Downarrow$

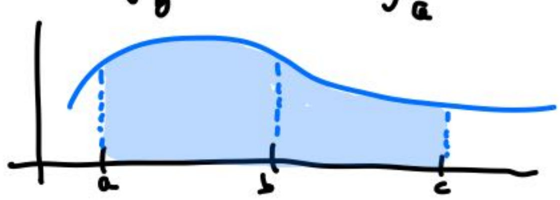
⑤ Comparison theorem: (again, think areas!)

$$f(x) \leq g(x) \text{ on } [a,b] \implies \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

⑥ $\int_a^b c\,dx = c\cdot(b-a)$

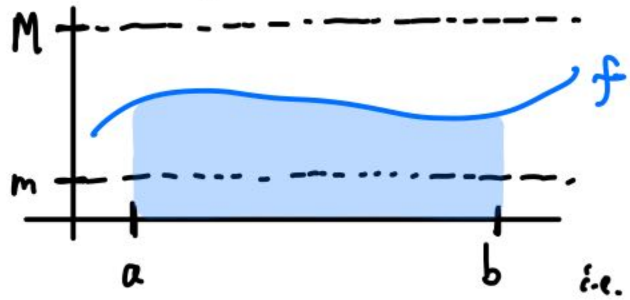


⑦ $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$



picture

Interesting application of ⑤ & ⑥ :



$$m \le f(x) \le M \text{ (on } [a,b])$$

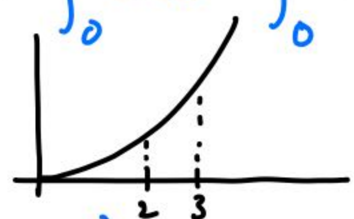
$$\Downarrow$$

$$\int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx$$

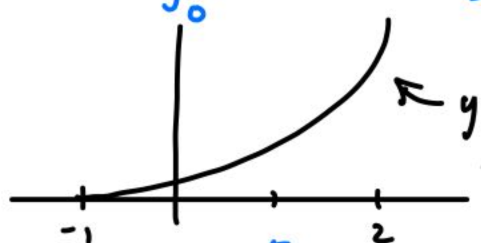
i.e. $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

Ex/ Given that $\int_0^a x^2\,dx = \frac{1}{3}a^3$, evaluate

$$\int_2^3 x^2\,dx = \int_0^3 x^2\,dx - \int_0^2 x^2\,dx = \frac{1}{3}3^3 - \frac{1}{3}2^3 = \frac{19}{3}$$



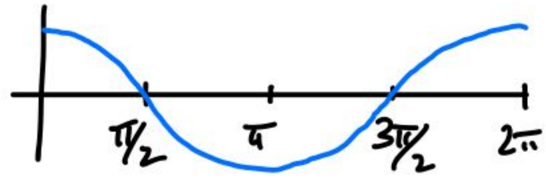
$$\int_{-1}^2 (x+1)^2\,dx = \int_0^3 x^2\,dx = \frac{1}{3}\cdot 3^3 = 9$$



$y=(x+1)^2$ is $y=x^2$ shifted one to the left

$$\int_0^5 2x^2\,dx = 2\int_0^5 x^2\,dx = 2\cdot\frac{1}{3}5^3 = \frac{250}{3}$$ //

Ex/ Find the value of $a \in [0, 2\pi]$ that maximizes $\int_0^a \cos(x)\,dx$.



Clearly $\pi/2$, by geometric reasoning! //