# Lecture 30: Some more series

Recall that given a sequence $\{a_n\}_{n=1}^{\infty}$ (or $\{a_n\}_{n=0}^{\infty}$), we can form the sequence of $n^{\text{th}}$ partial sums $\{S_n\}$:

$$S_n = \sum_{i=1}^{n} a_i \quad \left(\text{or } \sum_{i=0}^{n} a_i\right).$$

We then say that the infinite series $\sum_{n=1}^{\infty} a_n$ $\begin{cases}\text{converges}\\ \text{diverges}\end{cases}$ if the sequence $S_n$ $\begin{cases}\text{converges}\\ \text{diverges}\end{cases}$. In the convergent case, if $S := \lim_{n\to\infty} S_n$, then we write

$$\sum_{n=1 \,(\text{or } 0)}^{\infty} a_n = S,$$

and call $S$ the sum of the series.

Basic facts:

- If $\sum a_n$ and $\sum b_n$ both converge, then
$$\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n.$$
- If $\sum a_n$ converges, then $\sum c a_n = c \sum a_n$.
- Geometric series: if $|r| < 1$, $\sum r^n = \frac{1}{1-r}$; otherwise, $\sum r^n$ diverges.

I'll start with a few more examples, then turn to telescoping sums and the "test for divergence".

Ex 1/ The $n^{\text{th}}$ partial sum of a series is given by $S_n = \frac{n-1}{n+1}$. Find $a_n$, $\lim_{n\to\infty} a_n$, and $S$ (the sum of the series).

$$a_n = S_n - S_{n-1} = \frac{n-1}{n+1} - \frac{(n-1)-1}{(n-1)+1} = \frac{n-1}{n+1} - \frac{n-2}{n}$$

$$= \frac{n(n-1)-(n+1)(n-2)}{n(n+1)} = \frac{\cancel{n^2} - \cancel{n} - \cancel{n^2} + \cancel{n} + 2}{n(n+1)} = \frac{2}{n(n+1)}$$

$\Rightarrow \lim_{n\to\infty} a_n = 0$. Also, $S = \sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n-1}{n+1}$

$$= \lim_{n\to\infty} \frac{1-1/n}{1+1/n} = 1. \quad //$$

Ex 2/ Write the repeating decimal 0.999... as a fraction.

This is the sum of the geometric series $\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$

$$= \sum_{n=0}^{\infty} \frac{9}{10}\left(\frac{1}{10}\right)^n = \frac{9/10}{1-1/10} = \frac{9/10}{9/10} = 1 \quad !? \quad \text{Yes.}$$

Decimal representation of real numbers isn't unique. Terminating decimals (say 0.347) always have an alternate representation (as 0.3469999....). //

Ex 3/ (Keynesian "multiplier effect" on trade activity)

Idea: – gov't. spends £X (say, on computers: regard this as salaries of microchip manufacturers/etc.)

– recipients spend £c·X (on food, vacation, & indirectly on gov't projects thru taxation): c = "marginal propensity to consume"

– the remainder (£s·X) is saved, or "leaks" (debt repayments, imports, etc.); s = 1 − c.

– net effect of the gov't expenditure given by geometric series $X + cX + c^2X + \cdots = X/(1-c) = X/s$. //

**TRY:** For what values of $x$ does $\sum_{n=0}^{\infty} \frac{(x-1)^n}{2^n}$ converge?

## Telescoping sums

**Ex 4/** Consider the expression (series)

(1) $$\overset{a_1}{1} + \overbrace{(-1+1)}^{a_2} + \overbrace{(-1+1)}^{a_3} + \overbrace{(-1+1)}^{a_4} + \cdots$$

It is very tempting to cancel this way

$$\cancel{1+-1}+\cancel{1+-1}+\cancel{1+-1}+\cancel{1+\cdots} = 0?$$

and then this way

$$1+\cancel{-1+1}+\cancel{-1+1}+\cancel{-1+1}+\cdots = 1?$$

... which proves $0 = 1$, right? Or: start with

(2) $$\overset{a_1}{(-0+1)} + \overset{a_2}{(-1+2)} + \overset{a_3}{(-2+3)} + \overset{a_4}{(-3+4)} + \cdots$$

which on the one hand is $1+1+1+1+\cdots = \infty$, and on the other would appear to "telescope" as

$$-0\cancel{+1+(-1)}\cancel{+2+(-2)}\cancel{+3+(-3)}\cancel{+4+\cdots} = 0?$$

Obviously something must be wrong.

Recall the definition of "sum of a series": take the limit of the sequence of partial sums. For (1), the partial sums $S_n$ are all 1. For (2), the partial sum $S_n = \underbrace{1+\cdots+1}_{n \text{ times}} = n$.

So the sum of (1) is 1, and the sum of (2) is $\infty$.

The moral of this example is to pay attention to definitions, and to be careful with "rearranging" series. //

Ex 5/ Compute $\sum_{n=1}^{\infty}\left(\sin\left(\frac{1}{n}\right)-\sin\left(\frac{1}{n+1}\right)\right)$.

The nth partial sum is $S_n =$

$$\left(\sin(1)-\cancel{\sin\left(\tfrac{1}{2}\right)}\right)+\left(\cancel{\sin\left(\tfrac{1}{2}\right)}-\cancel{\sin\left(\tfrac{1}{3}\right)}\right)+\left(\cancel{\sin\left(\tfrac{1}{3}\right)}-\cancel{\sin\left(\tfrac{1}{4}\right)}\right)+\cdots+\left(\cancel{\sin\left(\tfrac{1}{n}\right)}-\sin\left(\tfrac{1}{n+1}\right)\right)$$

$= \sin(1) - \sin\left(\frac{1}{n+1}\right)$. So the sum of the series is

$\lim_{n\to\infty} S_n = \sin(1) - \lim_{n\to\infty}\sin\left(\frac{1}{n+1}\right) = \sin(1) - \sin(0) = \sin(1)$. //

Ex 6/ Compute $\sum_{n=1}^{\infty}\left(\left(\frac{1}{6}\right)^n - \left(\frac{1}{6}\right)^{n+1}\right)$.

The same technique yields $S_n = \frac{1}{6} - \left(\frac{1}{6}\right)^{n+1} \longrightarrow \frac{1}{6}$.

Or, you can write $\sum\left(\frac{1}{6}\right)^n - \sum\left(\frac{1}{6}\right)^{n+1} = \frac{1/6}{1-1/6} - \frac{1/36}{1-1/6}$

$= \frac{1}{5} - \frac{1}{30} = \frac{1}{6}$.

WARNING: You cannot write $\sum\left(\sin\left(\frac{1}{n}\right) - \sin\left(\frac{1}{n+1}\right)\right) = \sum\sin\left(\frac{1}{n}\right) - \sum\sin\left(\frac{1}{n+1}\right)$ because $\sum\sin\left(\frac{1}{n}\right)$ and $\sum\sin\left(\frac{1}{n+1}\right)$ don't converge. //

Ex 7/ Compute $\sum_{n=1}^{\infty}\frac{1}{n^2+2n}$.

$\sum_{n=1}^{\infty}\frac{1}{n(n+2)} = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right)$ has $n^{\text{th}}$ partial sum $S_n =$

$$\frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+2}\right) = \frac{1}{2}\left\{\left(1-\cancel{\tfrac{1}{3}}\right)+\left(\tfrac{1}{2}-\cancel{\tfrac{1}{4}}\right)+\left(\cancel{\tfrac{1}{3}}-\cancel{\tfrac{1}{5}}\right)+\cdots+\left(\cancel{\tfrac{1}{n-1}}-\tfrac{1}{n+1}\right)+\left(\cancel{\tfrac{1}{n}}-\tfrac{1}{n+2}\right)\right\}$$

$$= \frac{1}{2}\left\{1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right\} \xrightarrow[n\to\infty]{\lim} \frac{1}{2}\left\{1+\frac{1}{2}\right\} = \frac{3}{4}.$$

Once again, you cannot write $\sum\left(\frac{1}{n}-\frac{1}{n+2}\right) = \sum\frac{1}{n} - \sum\frac{1}{n+2}$, because $\sum\frac{1}{n}$ and $\sum\frac{1}{n+2}$ diverge. //

$\boxed{\text{TRY: Compute } \sum_{n=2}^{\infty} \frac{1}{n(n+1)}}$  [Ans: $\sum_{n=2}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=\frac{1}{2}$.]

## A "Divergence Test"

- If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Why? $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow$ $S_n = \sum_{i=1}^{n} a_n$ converges (say, to $S$).

So $a_n = S_n - S_{n-1}$ limits to $S - S = 0$.

Therefore:

- $\boxed{\text{If } \{a_n\} \text{ diverges or } \lim_{n\to\infty} a_n \neq 0, \text{ then } \sum_{n=1}^{\infty} a_n \text{ diverges.}}$

Ex 8/ Does $\sum_{n=1}^{\infty} \underbrace{\ln\left(\frac{n^2+1}{2n^2+1}\right)}_{a_n}$ converge or diverge?

Since $\lim_{n\to\infty} a_n = \ln\left(\lim_{n\to\infty}\frac{n^2+1}{2n^2+1}\right) = \ln\left(\lim_{n\to\infty}\frac{1+n^{-2}}{2+n^{-2}}\right) = \ln\left(\frac{1}{2}\right) \neq 0$,

$\sum a_n$ <u>diverges</u>.

//

WARNING: The converse of the divergence test is <u>false</u>.

For instance, $\lim_{n\to\infty}\frac{1}{n} = 0$, but $\sum \frac{1}{n}$ diverges nevertheless.