Lecture 31: The Integral Test

Today we'll explore the relationship between sums of series and improper integrals: to test for convergence or divergence, and (in the event of convergence) to approximate the sum.

We've already seen that \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges: the partial sum

\[
S_2^n = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \cdots + \left(\frac{1}{(m-1)+1} + \cdots + \frac{1}{2^m}\right)
\]


\[
> 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}
\]

and so given \( M \), taking \( N \) at least \( 2^{2^{n-2}} \) makes \( S_n \geq M \) for any \( n \geq N \). So \( \lim_{n \to \infty} S_n = \infty \Rightarrow \sum \frac{1}{n} = \infty \).

But there's another, more geometrically appealing, way:

in the picture, \( S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{k} \) is the sum of the areas of the (red) boxes. The (blue) area under the curve is \( \int_{1}^{n+1} \frac{dx}{x} \). So we have \( S_n > \int_{1}^{n+1} \frac{dx}{x} \), hence

\[
\sum_{n=1}^{\infty} \frac{1}{n} = \lim_{n \to \infty} S_n > \lim_{n \to \infty} \int_{1}^{n+1} \frac{dx}{x} = \int_{1}^{\infty} \frac{dx}{x}.
\]

That is, if \( \int_{1}^{\infty} \frac{dx}{x} \) is \( \infty \), then \( \sum \frac{1}{n} \) certainly is. (More precisely, if the right-hand limit diverges to \( \infty \) — which \( \int_{1}^{\infty} \frac{dx}{x} = \ln(b) - \ln(a) = \ln(b/a) \) — so does the left-hand one.) This generalizes to...
The Integral Test: If \( a_n = f(n) \), where \( f \) is positive, continuous, and decreasing for \( x \geq 1 \), then \( \int_1^\infty f(x) \, dx \) and \( \sum_{n=1}^\infty a_n \) are partners in crime: either both diverge or both converge.

Ex 1/ For what values of \( p > 0 \) does the “\( p \)-series”

\[
\sum_{n=1}^\infty \frac{1}{n^p}
\]

converge/diverge?

We consider \( \int_1^\infty \frac{dx}{x^p} = \lim_{b \to \infty} \int_1^b \frac{dx}{x^p} = \lim_{b \to \infty} \frac{b^{1-p} - 1}{1-p} \)

\[= \lim_{b \to \infty} \left( \frac{b}{b^p} - \frac{1}{b^{p-1}} \right) = \begin{cases} \infty, & \text{if } p < 1 \\ \frac{1}{p-1}, & \text{if } p > 1 \end{cases} \]

\( \frac{1}{x^p} \) is continuous, positive, and decreasing on \( [1, \infty) \), so by the Integral Test, \( \sum \frac{1}{n^p} \) converges for \( p > 1 \) and diverges for \( p \leq 1 \).

Remark: The Integral Test has a slight generalization which can be useful in some cases: only assume \( f \) is \( > 0 \), continuous, and decreasing for \( n \geq k \). Then \( \int_k^{\infty} f(x) \, dx \) and \( \sum_{n=k}^\infty a_n \) converge or diverge together; and \( \sum_{n=k}^\infty a_n \) converges/diverges with \( \int_k^{\infty} f(x) \, dx \) (why?).

Ex 2/ Does \( \sum_{n=1}^\infty \frac{n^2}{n!} \) converge or diverge?

Well, \( a_n = f(n) \) where \( f(x) = xe^{-x/3} \). This is \( > 0 \) and continuous for all \( x > 0 \), but \( f'(x) = (1 - \frac{x^2}{9})e^{-x/3} \) is negative only once \( x > 3 \). So take \( k = 4 \):

\[
\int_4^\infty xe^{-x/3} \, dx = 9e^{-x/3} \bigg|_4^\infty = 9e^{-\infty} - 36e^{-4/3} < 0
\]

converges; \( \therefore \) so does the \( \sum \).
Why does the Integral Test work? We have \( a_n = f(n) \).

1. Shows \( a_k + \cdots + a_n \geq \int_k^{n+1} f(x) \, dx \)

\[
\lim_{n \to \infty} \sum_{k=n}^{n+1} a_n \geq \int_k^{n+1} f(x) \, dx \Rightarrow \sum_{n=k}^{\infty} a_n \geq \int_k^{\infty} f(x) \, dx \Rightarrow \text{if } \int = \infty, \sum = \infty.
\]

2. Shows \( a_k + \cdots + a_{n+1} \leq \int_k^{n+1} f(x) \, dx \)

\[
\Rightarrow a_k + \cdots + a_{n+1} \leq a_k + \int_k^{n+1} f(x) \, dx \Rightarrow \sum_{n=k}^{\infty} a_n \leq a_k + \int_k^{\infty} f(x) \, dx \Rightarrow \text{if } \int \text{ finite, } \sum \text{ finite.}
\]

(More precisely: \( S_n = a_k + \cdots + a_n \) is monotonically increasing since \( a_n < f(n) > 0 \), and bounded above by \( a_k + \int_k^{\infty} f(x) \, dx \). So it converges.)

Try: Which diverges?

\[
\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}, \sum_{n=1}^{\infty} \frac{\ln n}{n^2}, \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \quad \text{or} \quad \sum_{n=1}^{\infty} \frac{1}{n \sqrt{n}}.
\]

[Ans: this one.]
Another way to state the inequalities in the picture is:

\[ \int_{t+1}^{\infty} f(x) \, dx \leq \sum_{i=t+1}^{\infty} s_i \leq \int_{t+1}^{\infty} f(x) \, dx \]

or preferably, substituting \( n \) for \( t \)

\[ \int_{n+1}^{\infty} f(x) \, dx \leq \sum_{i=n+1}^{\infty} a_i = S - S_n \leq \int_{n+1}^{\infty} f(x) \, dx \]

This gives an estimate on the remainder \( R_n = S - S_n \)

\[ \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} \bar{a}_i = \sum_{i=n+1}^{\infty} \bar{a}_i , \]

and an answer to the question "how good an approximation to \( S = \sum_{i=1}^{\infty} a_i \) is \( S_n \)?"

Actually, we can get a much better approximation to \( S \) without summing up more terms in the series; from (27) we have

\[ S_n + \int_{n+1}^{\infty} f(x) \, dx \leq S \leq S_n + \int_{n+1}^{\infty} f(x) \, dx . \]

So one might consider

\[ S_n := S_n + \frac{\int_{n+1}^{\infty} f(x) \, dx + \int_{n}^{\infty} f(x) \, dx}{2} \]

as an approximation to \( S \). Since \( S_n \) is the midpoint of the interval in (27), it has to be within half its width of \( S \). That width is \( \int_{n+1}^{\infty} f(x) \, dx - \int_{n}^{\infty} f(x) \, dx = \int_{n}^{n+1} f(x) \, dx \), so

\[ |S - S_n| \leq \frac{1}{2} \int_{n}^{n+1} f(x) \, dx , \]

a lot better than (27)!
**Ex 3**/ The Riemann $\zeta$-function is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$  

The "even" $\zeta$-values $\zeta(2n)$ are rational multiples of $\pi^{2n}$, while the odd ones are far more mysterious (we expect $\zeta(3)$ has no algebraic relationship with $\pi$, but all we know is it's irrational). How good are $S_5$ and $S_5$ as approximations of $\zeta(2)$?

So, $\zeta(2) = \frac{\pi^2}{6}$ by a famous result of Euler. That's $\approx 1.644934$. 

$S_5 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} \approx 1.463611$. But $S_5$ is

$$S_5 + \frac{1}{2} \left( \int_0^1 \frac{\text{d}x}{x^2} + \int_0^1 \frac{\text{d}x}{x^2} \right) = S_5 + \frac{1}{2} \left( \frac{1}{5} + \frac{1}{6} \right) = S_5 + \frac{11}{60} \approx 1.676944.$$

Clearly a lot better; and what's more, even if you didn't know $\zeta(2)$, you'd know $S_5$ was within $\frac{1}{2} \int_0^1 \frac{dx}{x^2} = \frac{1}{2} (1 - \frac{1}{2}) = \frac{1}{4}$ of it.

---

**Try:** How large must $n$ be for $S_n$ to be correct to 5 decimal places? for $S_n$? (we're still taking as our sum $\sum \frac{1}{n^2}$)

**Ans:** 5 decimal places means within $0.00005 = \frac{5}{10^5}$. 

Since $S - S_n \leq \int_{n+1}^{\infty} \frac{dx}{x^2} = \frac{1}{n}$, you'd need $n \geq \frac{10^5}{5} = 200000$ for $S_n$. But for $S_n$, we only need to make $\frac{1}{2} \int_0^1 \frac{\text{d}x}{x^2} = \frac{1}{2} (1 - \frac{1}{n+1}) = \frac{1}{2n(n+1)} < \frac{5}{10^5}$, and this is satisfied for $n \geq 316.$