Lecture 31 : The Integral Test

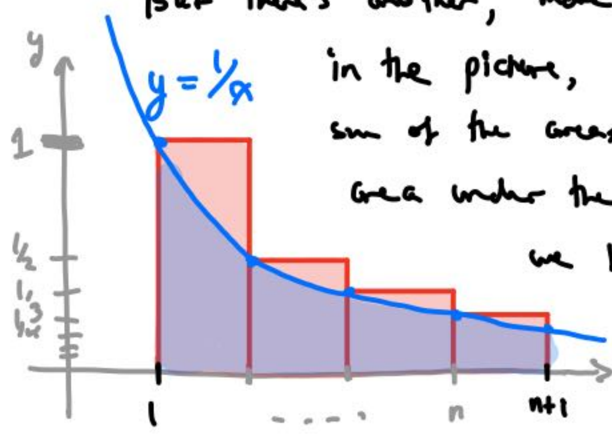
Today we'll explore the relationship between sums of series and improper integrals: to test for convergence or divergence, and (in the event of convergence) to approximate the sum.

We've already seen that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges: the partial sum

$$S_{2^m} = 1 + \frac{1}{2} + \left(\frac{1}{3}+\frac{1}{4}\right) + \left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right) + \cdots + \underbrace{\left(\frac{1}{2^{m-1}+1} + \cdots + \frac{1}{2^m}\right)}_{2^{m-1}\text{ terms}}$$
$$> 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}$$
$$= 1 + \frac{m}{2}$$

and so given M, taking N at least 2^{2M-2} makes $S_n \geq M$ for any $n \geq N$. So $\lim_{n\to\infty} S_n = \infty \Rightarrow \sum \frac{1}{n} = \infty$.

But there's another, more geometrically appealing, way:



in the picture, $S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is the sum of the areas of the (red) boxes. The (blue) area under the curve is $\int_1^{n+1} \frac{dx}{x}$. So we have $S_n > \int_1^{n+1} \frac{dx}{x}$, hence

$$\text{“} \sum_{n=1}^{\infty} \frac{1}{n} = \lim_{n\to\infty} S_n > \lim_{n\to\infty} \int_1^{n+1} \frac{dx}{x} = \int_1^{\infty} \frac{dx}{x}. \text{”}$$

That is, if $\int_1^{\infty} \frac{dx}{x}$ is ∞, then $\sum \frac{1}{n}$ certainly is. (More precisely, if the right-hand limit diverges to ∞ — which $\int_1^b \frac{dx}{x} = \ln(b)$ <u>does</u> — so does the left-hand one.) This generalizes to...

THE INTEGRAL TEST: If $a_n = f(n)$, where f is positive, continuous, and decreasing for $x \geq 1$, then $\int_1^\infty f(x)\,dx$ and $\sum_{n=1}^\infty a_n$ are partners in crime: either both diverge or both converge.

Ex 1/ For what values of $p > 0$ does the "p-series" $\sum_{n=1}^\infty \frac{1}{n^p}$ converge / diverge?

$p \neq 1$ (already know it diverges)

We consider $\int_1^\infty \frac{dx}{x^p} = \lim_{b\to\infty} \int_1^b x^{-p}\,dx = \lim_{b\to\infty} \frac{x^{-p+1}}{-p+1}\Big|_1^b$

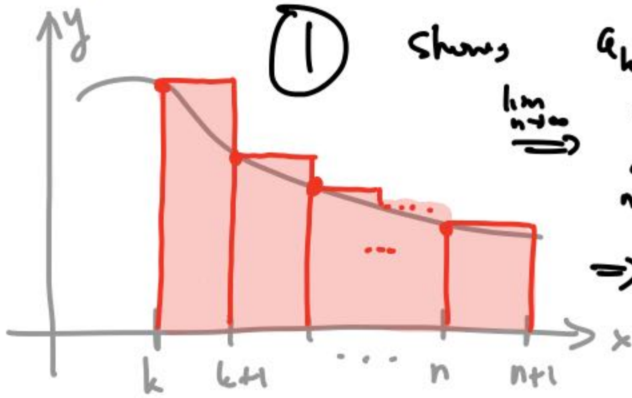
$= \lim_{b\to\infty}\left(\frac{b^{1-p}}{1-p} - \frac{1}{1-p}\right) = \begin{cases} \infty, & \text{if } p<1 \\ \frac{1}{p-1}, & \text{if } p>1 \end{cases}$. Certainly $\frac{1}{x^p}$ is continuous, positive, and decreasing on $[1,\infty)$, so by the Integral Test, $\sum \frac{1}{n^p} \begin{cases} \text{converges for } p>1 \\ \text{diverges for } p \leq 1. \end{cases}$ //

Remark: The Integral Test has a slight generalization which can be useful in some cases: only assume f is >0, continuous, and decreasing for $n \geq k$. Then $\int_k^\infty f(x)\,dx$ and $\sum_{n=k}^\infty a_n$ converge & diverge together; and $\sum_{n=k}^\infty$ converges/diverges with $\sum_{n=1}^\infty$ (why?).

Ex 2/ Does $\sum_{n=1}^\infty n e^{-\frac{n^2}{18}}$ converge or diverge?

Well, $a_n = f(n)$ where $f(x) = x e^{-x^2/18}$. This is >0 & continuous for all $x>0$, but $f'(x) = \left(1 - \frac{x^2}{9}\right) e^{-x^2/18}$ is negative only once $x > 3$. So take $k = 4$: $\int_4^\infty x e^{-x^2/18}\,dx = -9e^{-x^2/18}\Big|_4^\infty = 9e^{-16/18}$ converges; $\therefore$ so does the $\sum$. //

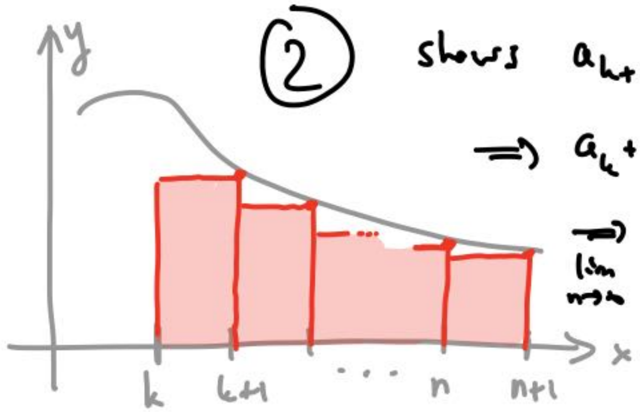
Why does the Integral Test work? We have $a_n = f(n)$.



① shows $a_k + \cdots + a_n \geq \int_k^{n+1} f(x)\,dx$

$\xRightarrow{\lim_{n\to\infty}} \sum_{n=k}^{\infty} a_n \geq \int_k^{\infty} f(x)\,dx$.

$\Rightarrow$ if $\int = \infty$, $\sum = \infty$.



② shows $a_{k+1} + \cdots + a_{n+1} \leq \int_k^{n+1} f(x)\,dx$

$\Rightarrow a_k + \cdots + a_{n+1} \leq a_k + \int_k^{n+1} f(x)\,dx$

$\xRightarrow{\lim_{n\to\infty}} \sum_{n=k}^{\infty} a_n \leq a_k + \int_k^{\infty} f(x)\,dx$

$\Rightarrow$ if $\int$ finite, $\sum$ finite.

(More precisely: $S_n = a_k + \cdots + a_n$ is monotonically increasing since $a_n = f(n) > 0$, and bounded above by $a_k + \int_k^{\infty} f(x)\,dx$. So it converges.)

Try: Which diverges —

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}, \quad \sum_{n=1}^{\infty} \frac{\ln n}{n^2}, \quad \sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$$

or $\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}}$? [Ans: this one.]

Another way to state the inequalities in the picture is:
$$\int_{k+1}^{\infty} f(x)\,dx \le \sum_{i=k+1}^{\infty} a_i \le \int_k^{\infty} f(x)\,dx$$
or preferably, substituting n for k

sum of the series

nth partial sum

$$(*) \quad \boxed{\int_{n+1}^{\infty} f(x)\,dx \le \sum_{i=n+1}^{\infty} a_i = S - S_n \le \int_n^{\infty} f(x)\,dx}$$

This gives an estimate on the remainder $R_n = S - S_n = \sum_{i=1}^{\infty} a_i - \sum_{i=1}^{n} a_i = \sum_{i=n+1}^{\infty} a_i$, and an answer to the question "how good an approximation to $S = \sum_{i=1}^{\infty}$ is S_n ?"

Actually, we can get a much better approximation to S without summing up more terms in the series: from $(*)$ we have
$$S_n + \int_{n+1}^{\infty} f(x)\,dx \le S \le S_n + \int_n^{\infty} f(x)\,dx.$$
So one might consider
$$\mathcal{S}_n := S_n + \frac{\int_n^{\infty} f(x)\,dx + \int_{n+1}^{\infty} f(x)\,dx}{2}$$
as an approximation to S. Since $\mathcal{S}_n$ is the midpoint of the interval in $(*)$, it has to be within half its width of S. That width is $\int_n^{\infty} f\,dx - \int_{n+1}^{\infty} f\,dx = \int_n^{n+1} f\,dx$, so
$$\left| S - \mathcal{S}_n \right| \le \frac{1}{2} \int_n^{n+1} f(x)\,dx,$$
a lot better than $(*)$!

Ex 3/ The Riemann ζ-function is given by $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$. The "even" ζ-values $\zeta(2m)$ are rational multiples of π^{2m}, while the odd ones are far more mysterious (we expect $\zeta(3)$ has no algebraic relationship with π, but all we know is it's irrational).

How good are S_5 and $\mathcal{S}_5$ as approximations of $\zeta(2)$?

So, $\zeta(2) = \frac{\pi^2}{6}$ by a famous result of Euler. That's ≈ 1.644934.

$S_5 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} \approx 1.463611$. But $\mathcal{S}_5$ is

$S_5 + \frac{1}{2}\left(\int_5^\infty \frac{dx}{x^2} + \int_6^\infty \frac{dx}{x^2}\right) = S_5 + \frac{1}{2}\left(\frac{1}{5} + \frac{1}{6}\right) = S_5 + \frac{11}{60} \approx 1.646944$.

Clearly a lot better; and what's more, even if you didn't know $\zeta(2)$, you'd know $\mathcal{S}_5$ was within $\frac{1}{2}\int_5^6 \frac{dx}{x^2} = \frac{1}{2}\left(\frac{1}{5} - \frac{1}{6}\right) = \frac{1}{60}$ of it. //

TRY: How large must n be for S_n to be correct to 5 decimal places? for $\mathcal{S}_n$? (We're still taking as our sum $\sum \frac{1}{n^2}$)

Ans: 5 decimal places means within $0.000005 = \frac{5}{10^6}$.

Since $S - S_n \leq \int_n^\infty \frac{dx}{x^2} = \frac{1}{n}$, you'd need $n \geq \frac{10^6}{5} = 200000$ for S_n. But for $\mathcal{S}_n$, we only need to make $\frac{1}{2}\int_n^{n+1} \frac{dx}{x^2} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{2n(n+1)} < \frac{5}{10^6}$, and this is satisfied for $n \geq 316$.