Lecture 32: Comparison tests

In the last lecture, we compared series to improper integrals, and discussed using the integrals to improve the estimates of the sum of the series given by its partial sums.
Today we're going to compare series with each other: if given a series $\sum a_n$ we regard each a_n as a puff of air in blowing up a balloon, and $\sum b_n$ as a series of larger puffs ($b_n \geq a_n \geq 0$), then we have the picture

If the inner balloon eventually pops (i.e. the series diverges), so will the outer one; if the outer one never pops (i.e. the series converges), the inner one won't pop either. But if the outer one pops (or the inner one doesn't), that won't tell us anything about the other.

Why is this useful? We have completely analyzed convergence & divergence of two types of series:

- geometric series $\sum r^n$ (conv. if $-1 < r < 1$, div. otherwise)
- p-series $\sum \frac{1}{n^p}$ (conv. if $p > 1$, div. otherwise)

and these will now provide standards against which we can measure other series.

BASIC COMPARISON TEST If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are positive-term series, and $b_n \geq a_n$ for all $n \geq N$, then:

- $\sum b_n$ converges $\Longrightarrow$ $\sum a_n$ converges
$\Downarrow$
- $\sum a_n$ diverges $\Longrightarrow$ $\sum b_n$ diverges

[Why is it true? If $\sum b_n = B$ converges, then B is an upper bound on the (increasing) partial sums S_n of $\sum a_n$, which therefore converge. The second statement is the contrapositive of the first.]

Ex 1/ Does $\sum_{n=1}^{\infty} \frac{n}{5n^2-4}$ converge or diverge?

Guess: diverges, since sort of like $\frac{1}{5n}$ for large n?

Correct: $5n^2 > 5n^2 - 4 \Rightarrow \underbrace{\frac{n}{5n^2-4}}_{b_n} > \frac{n}{5n^2} = \frac{1}{5}\cdot\frac{1}{n} = a_n$

and $\sum a_n = \frac{1}{5}\sum \frac{1}{n}$ diverges. //

Ex 2/ Does $\sum_{n=1}^{\infty} \frac{n}{2^n(n+1)}$ converge or diverge?

Guess: like $\frac{1}{2^n}$, so converges.

Again correct! $\frac{n}{n+1} < 1 \Rightarrow \underbrace{\frac{n}{2^n(n+1)}}_{a_n} < \frac{1}{2^n} = b_n$ and $\sum b_n$ converges

$\Rightarrow \sum a_n$ converges. //

Ex 3/ Does $\sum_{n=1}^{\infty} \frac{1}{n^2-4n+5}$ converge or diverge?

Guess: like $\frac{1}{n^2}$ for large n? But $\frac{1}{n^2-4n+5} > \frac{1}{n^2}$ (for $n \geq 2$), which goes the wrong way for what we want. On the other hand,
$(n^2-4n+5) - \frac{n^2}{2} = \frac{n^2}{2} - 4n + 5 \geq 0$ for $n \geq 7$, so
$\frac{1}{n^2-4n+5} \leq \frac{2}{n^2}$ for $n \geq N := 7$, and since $\sum \frac{2}{n^2}$ converges, so does our series. //

TRY: Which of the following converges?

$$\sum_{n=1}^{\infty}\frac{1}{2+3^n},\quad \sum_{n=2}^{\infty}\frac{1}{\sqrt{n}-1},\quad \sum_{n=1}^{\infty}\frac{1}{n+1},\quad \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+1}.$$

Ans. The first one. But to show the last diverges, we have to work: $\frac{1}{\sqrt{n}+1} < \frac{1}{\sqrt{n}}$, not the direction we want. But $\sqrt{n}+1 \le 2\sqrt{n} \Rightarrow \frac{1}{\sqrt{n}+1} \ge \frac{1}{2\sqrt{n}}$ and $\sum \frac{1}{2\sqrt{n}} = \frac{1}{2}\sum\frac{1}{n^{1/2}}$ diverges $\Rightarrow \sum\frac{1}{\sqrt{n}+1}$ converges.

How do we avoid the contortions with inequalities as in Ex. 3 and $\sum \frac{1}{\sqrt{n}+1}$? After all, $\frac{1}{n^2-4n+5}$ and $\frac{1}{n^2}$, or $\frac{1}{\sqrt{n}+1}$ and $\frac{1}{\sqrt{n}}$, are approximately the same size for large n, so it shouldn't be as hard as it was. That's why we need this:

LIMIT COMPARISON TEST If $\sum a_n$ and $\sum b_n$ are positive-term series, and $\lim_{n\to\infty}\frac{a_n}{b_n} = L$ is positive (not ∞), then the two series converge or diverge together. (If $L=0$ and $\sum b_n$ converges, then $\sum a_n$ converges.)

Why is it true? Well, since $\frac{a_n}{b_n} \to L$, there's an N such that $n \ge N \Rightarrow \left|\frac{a_n}{b_n} - L\right| < \frac{L}{2} \Rightarrow \frac{L}{2} < \frac{a_n}{b_n} < \frac{3L}{2}$
$\Rightarrow a_n < \frac{3L}{2} b_n$ and $b_n < \frac{2}{L} a_n$. If $\sum b_n$ converges, then $\sum \frac{3L}{2} b_n$ converges $\xRightarrow[\text{comparison test}]{\text{basic}}$ $\sum a_n$ converges. If $\sum a_n$ converges, then $\sum \frac{2}{L} a_n$ converges $\xRightarrow[\text{c.t.}]{\text{basic}}$ $\sum b_n$ converges.
(If $L=0$, then $b_n > a_n$ eventually, so again done by basic c.t.)

Returning to $a_n = \frac{1}{n^2-4n+5}$, take $b_n = \frac{1}{n^2}$. Then

$$L := \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2/n^2}{(n^2-4n+5)/n^2} = \lim_{n\to\infty} \frac{1}{1-4n^{-1}+5n^{-2}} = \frac{1}{1-0+0} = 1.$$

Since $\sum b_n$ converges, so does $\sum a_n$.

Ex 4/ Does $\sum_{n=1}^{\infty} \frac{3n-2}{n^3-2n^2+11}$ converge or diverge?

Compare against $b_n = \frac{3}{n^2}$:

$$L := \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(3n-2)\,n^2}{(n^3-2n^2+11)3} = \lim_{n\to\infty} \frac{(3n^3-2n^2)/n^3}{(3n^3-6n^2+33)/n^3}$$

$$= \lim_{n\to\infty} \frac{3-2n^{-1}}{3-6n^{-1}+33n^{-3}} = \frac{3-0}{3-0+0} = 1.$$

So $\sum b_n$ conv. $\Rightarrow$ $\sum a_n$ conv. //

$\boxed{\text{TRY : Does } \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+19n}} \text{ converge or diverge?}}$

Ex 5/ Does $\sum_{n=1}^{\infty} \underbrace{\frac{\ln(n)}{n^2}}_{a_n}$ converge or diverge?

What should we compare with?

$b_n = \frac{1}{n^2} \rightsquigarrow L = \lim_{n\to\infty} \frac{a_n}{b_n} = \ln(n) = \infty$ ✗ (test fails)

$b_n = \frac{1}{n} \rightsquigarrow L = \lim_{n\to\infty} \frac{\ln(n)}{n} = 0$ (and $\sum b_n$ doesn't converge) ✗

$b_n = \frac{1}{n^{3/2}} \rightsquigarrow L = \lim_{n\to\infty} \frac{\ln(n)}{\sqrt{n}} = 0$ (and $\sum b_n$ converges)

$\Rightarrow \sum a_n$ converges. //

There is one more test I'll introduce now, just so you see it in this simpler (positive-term series) context before we do it in the context of absolute convergence. It isn't used in WeBWork 11 or the upcoming exam.

$\boxed{\text{RATIO TEST}}$ Let $\sum a_n$ be a positive-term series, with "limiting ratio" $\rho := \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

- $\underset{(\text{incl. } 0)}{\rho < 1} \implies \sum a_n$ converges
- $\underset{(\text{incl. } \infty)}{\rho > 1} \implies \sum a_n$ diverges
- $\rho = 1 \implies$ inconclusive

Why is it true? The idea is that $\sum a_n$ is "like" a geometric series with ratio ρ (though not enough like one that we can say anything when $\rho = 1$). For example, if $\rho < 1$, then taking $r \in (\rho, 1)$, and N large enough that $n \geq N \implies \frac{a_{n+1}}{a_n} < r$, we have $a_{N+k} < r\, a_{N+k-1} < \cdots < r^k a_N$. Thus $\sum_{n=N}^{\infty} a_n \leq \sum_{n=N}^{\infty} a_N r^{n-N} = \frac{a_N}{r^N} \sum_{n=N}^{\infty} r^n$ which converges (as $r<1$). So $\sum a_n$ converges, we're done by the basic comparison test.

Ex 6 / Does $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converge or diverge?

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}/(n+1)!}{2^n/n!} = \lim_{n\to\infty} \frac{\cancel{n!}}{\underbrace{\cancel{(n+1)!}}_{n+1}} \cdot \frac{2^{\cancel{n+1}}}{\cancel{2^n}}$$

$$= \lim_{n\to\infty} \frac{2}{n+1} = 0 \implies \text{converges.} \quad //$$

That was just a preview — we'll revisit this test in a week.