Lecture 33: Exam III Review

<u>Format</u>: 10 multiple choice problems

2 handgraded (multipart) problems

worth 15% of grade

<u>Material covered</u>: §§7.8, 8.1,3,5, and 11.1–4.

(I) Improper integrals

- $\int_a^b$, where $[a,b]$ contains a vertical asymptote
- $\int_a^\infty$, $\int_{-\infty}^b$, or $\int_{-\infty}^\infty$ (infinite intervals)

(II) Applications of integrals

- arclength
- area of a surface of revolution
- probability: density functions, median & mean

(III) Sequences

- basic properties & examples (like rational functions of n)
- use of L'Hôpital for indeterminate forms ($\frac{0}{0}, \frac{\infty}{\infty}, 0\cdot\infty, \infty-\infty$, and $0^0, \infty^0, 1^\infty$)
- squeeze theorem
- monotonic sequence theorem & use of induction
- recursively defined sequences

(IV) Series

- geometric series (conv./div., computation of sum)
- p-series (conv./div.)
- telescoping sums
- integral test and related inequalities } positive-term series
- comparison test ("basic" and "limit") }

(no error estimates)

The handgraded problems are entirely on (III)–(IV), though of course the integral test involves computing improper integrals.

Before turning to some examples, we discuss (III) a bit.

The Monotone Sequence Thm. says that an increasing sequence that is bounded above, or a decreasing sequence that is bounded below, is convergent. How do we check monotonicity & boundedness? Often this requires the use of the Principle of Mathematical Induction:

(*) Let $Q(m)$ be a statement about any $m \in \mathbb{N}$. If $Q(1)$ holds, and "$Q(n)$ is true if $Q(m)$ holds for all $m < n$", then $Q(n)$ holds for every n.

[This works because if $\mathcal{S} = \{x \in \mathbb{N} \mid Q(x) \text{ fails}\}$ were nonempty, it has a least element ℓ $(\neq 1)$. But then $Q(x)$ holds for every $x < \ell$, and thus $Q(\ell)$ holds, a contradiction. So $\mathcal{S}$ is empty.]

Let's see how this is used in an actual example.

Ex 1/ (TRY) $a_1 = 4$, $a_{n+1} = 4\sqrt{a_n} - 3$. What is the limit?

The first few terms are $4, 5, 4\sqrt{5} - 3, \ldots$ which appears to be increasing. Is it also bounded above?

- Say $Q(m)$ means "$a_m \le 100$". Clearly true for $m=1$. Suppose true for $m<n$. Then $a_n = 4\sqrt{a_{n-1}} - 3 < 4\sqrt{100} - 3 = 37 < 100$. By induction, $Q(n)$ holds for every n, i.e. $\{a_n\}$ is bounded above.
- Let $Q(m)$ stand for "$a_m < a_{m+1}$". True for $m=1$ $(4<5)$. Suppose true for $m<n$, in particular $a_{n-1} < a_n$. So then $\sqrt{a_{n-1}} < \sqrt{a_n} \implies a_n = 4\sqrt{a_{n-1}} - 3 < 4\sqrt{a_n} - 3 = a_{n+1}$. By induction, $Q(n)$ holds for all n, i.e. $\{a_n\}$ is increasing.
- By the Monotonic Sequence Theorem, $L = \lim_{n\to\infty} a_n$ exists. Take limits on both sides of $a_{n+1} = 4\sqrt{a_n} - 3$ to obtain
$$L = 4\sqrt{L} - 3 \implies L - 4\sqrt{L} + 3 = 0 \underset{\text{quadratic eqn.}}{\implies} L = 1 \text{ or } 9.$$
Which is it? Well, $\{a_n\}$ starts from 4 and is increasing, so can't limit to 1. So $L = 9$. //

Ex 2/ Find $L = \lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n$.

This is an indeterminate form of type 1^∞. Take ln of both sides, then apply L'Hôpital.

$$\ln(L) = \lim_{n\to\infty} \ln\left(\left(1+\frac{a}{n}\right)^n\right) = \lim_{n\to\infty} n \cdot \ln\left(1+\frac{a}{n}\right)$$
$$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{a}{x}\right)}{\frac{1}{x}} \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{1+a/x} \cdot \cancel{-\frac{a}{x^2}}}{\cancel{-\frac{1}{x^2}}}$$
$$= \lim_{x\to\infty} \frac{a}{1+\frac{a}{x}} = \frac{a}{1 + \cancel{\lim_{x\to\infty} \frac{a}{x}}^{\,0}} = a$$
$$\implies L = e^{\ln(L)} = e^a.$$
//

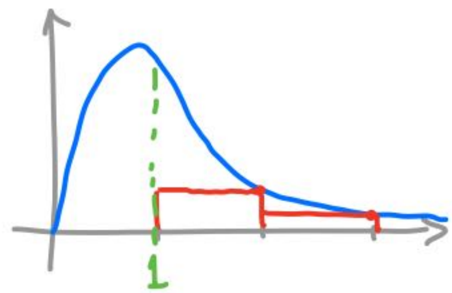
Turning to series:

Ex 3/ Compute $\sum_{n=1}^{\infty} \frac{3^{n+1}}{5^{n-1}}$.

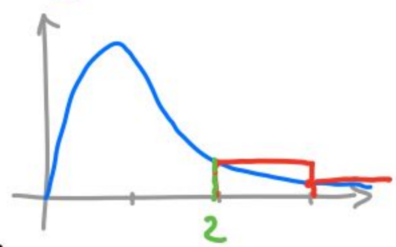
This is a geometric series with ratio $r = \frac{a_{n+1}}{a_n} = \frac{3^{n+2}/5^{n}}{3^{n+1}/5^{n-1}} = \frac{3}{5}$
and initial term $3^{1+1}/5^{1-1} = 9$. Since $-1 < \frac{3}{5} < 1$, the series converges, with sum $\frac{9}{1-3/5} = \frac{9}{2/5} = \frac{45}{2}$. //

Ex 4 / Which of $a = 0, 1, 2, 3$ gives the tightest but still
(TRY) correct inequality?: $\sum_{n=2}^{\infty} n^2 e^{-n^3} < \int_a^{\infty} x^2 e^{-x^3}\,dx$.

The integrand is decreasing (by taking derivative) for $x > \sqrt[3]{\frac{2}{3}}$.
So the picture is as shown, with $a=0$ or 1 giving a correct inequality.



On the other hand, if we look at $a = 2$ the inequality goes the other way, as you can see from this modification of the picture: the area under the curve is contained in the area under the boxes.



We also might ask: does the series converge?

Compute the integral! $\int_1^{\infty} x^2 e^{-x^3}\,dx = \lim_{b\to\infty} \int_1^b x^2 e^{-x^3}\,dx$
$= \lim_{b\to\infty} -\frac{1}{3} e^{-x^3} \Big|_1^b = \lim_{b\to\infty} \left(\frac{1}{3}e^{-1} - \frac{1}{3}e^{-b^3}\right) = \frac{1}{3e}$.

(How do I know that $e^{-b^3} \to 0$? It is $(e^{-1})^{b^3}$, where $e^{-1} < 1$ and $b^3 \to \infty$.) //

I'd also recommend looking over the past Participation problems on sequences & series.

For improper integrals, the trickier ones — though far less common — are the ones with a vertical asymptote.

Ex 5/ Determine which of $\int_{-2}^{1} \frac{dx}{x^2}$ & $\int_0^3 \frac{dx}{(x-1)^{2/3}}$ converges, & compute its value.

- $\int_{-2}^{1} \frac{dx}{x^2} = \lim_{b \to 0^-} \int_{-2}^{b} \frac{dx}{x^2} + \lim_{a \to 0^+} \int_a^1 \frac{dx}{x^2}$

$$= \lim_{b \to 0^-} \left(\frac{1}{-2} - \frac{1}{b}\right) + \lim_{a \to 0^+} \left(\frac{1}{a} - 1\right) \quad \underline{\underline{\text{diverges}}}$$

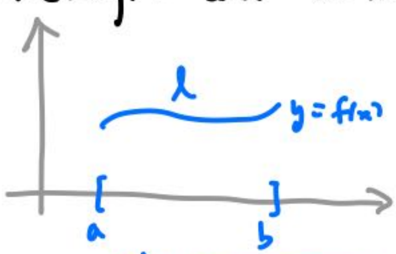
(if you ignore the asymptote, you'll get $\frac{1}{-2} - 1 = -\frac{3}{2}$, which is absurd)

- $\int_0^3 \frac{dx}{(x-1)^{2/3}} = \lim_{b \to 1^-} \int_0^b (x-1)^{-2/3}\,dx + \lim_{a \to 1^+} \int_a^3 (x-1)^{-2/3}\,dx$

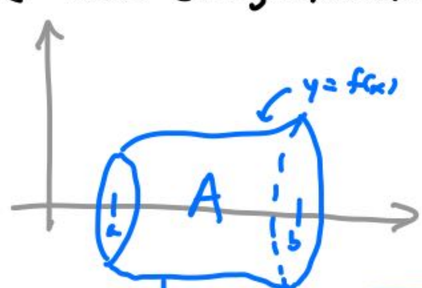
$$= \lim_{b \to 1^-} 3(x-1)^{1/3}\Big|_0^b + \lim_{a \to 1^+} 3(x-1)^{1/3}\Big|_a^3$$

$$= -3(-1)^{1/3} + 3(2)^{1/3} = 3(\sqrt[3]{2} + 1). \quad //$$

Arclength and surface area are more straightforward:



$l = \int_a^b \sqrt{1 + (f'(x))^2}\,dx$



$A = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2}\,dx$

Probability: a density function $f(x)$ is ≥ 0 and satisfies $\int_{-\infty}^{\infty} f(x)\,dx = 1$. (Sometimes you're asked to determine a constant to make this so.) The mean $\mu = \int_{-\infty}^{\infty} x f(x)\,dx$, and the median m satisfies $\frac{1}{2} = \int_m^{\infty} f(x)\,dx$. That's it.