Lecture 33: Exam III Review

Format: 10 multiple choice problems
2 hand graded (multipart) problems
worth 15% of grade

Material covered: 8.8, 7.8, 8.1, 35, and 11.1-4.

(I) Improper integrals
- \( \int_a^b \), where \([a,b]\) contains a vertical asymptote
- \( \int_a^\infty \), \( \int_0^b \), or \( \int_{-\infty}^b \) (infinite intervals)

(II) Applications of integrals
- arclength
- area of a surface of revolution
- probability: density functions, median & mean

(III) Sequences
- basic properties & examples (like rational functions of \( n \))
- use of L'Hôpital for indeterminate forms (0/0, \( \infty/\infty \), 0\( \cdot \)\( \infty \), \( \infty - \infty \), \( \infty 0 \), and \( 0^0 \), \( \infty^0 \), 1\(^\infty \))
- squeeze theorem
- monotonic sequence theorem & use of induction
- recursively defined sequences
(IV) Series
- geometric series (conv./div., computation & sum)
- $p$-series (conv. /div.)
- telescoping sums
- integral test and related inequalities
- comparison test ("hard" and "limit")

Positive-term series
(no error estimates)

The hand-graded problems are entirely on (III)-(IV), though of course the integral test involves computing improper integrals.

---

Before turning to some examples, we discuss (III) a bit.

The Monotone Sequence Theorem says that an increasing sequence that is bounded above, or a decreasing sequence that is bounded below, is convergent. How do we check monotonicity & boundedness? Often this requires the use of the Principle of Mathematical Induction:

Let $Q(m)$ be a statement about any $m \in \mathbb{N}$. If $Q(1)$ holds, and $Q(m)$ is true if $Q(m)$ holds for all $m < n$, then $Q(n)$ holds for every $n$.

This works because if $S = \{x \in \mathbb{N} \mid Q(x) \text{ fails}\}$ were nonempty, it has a least element $l (\neq 1)$. But then $Q(l)$ holds for every $x < l$, and thus $Q(l)$ holds, a contradiction. So $S$ is empty.

Let's see how this is used in an actual example.

**Ex 1**
- $a_1 = 4$, $a_{n+1} = 4\sqrt{a_n} - 3$. What is the limit?
- The first few terms are 4, 5, $4\sqrt{5} - 3$, ... which appears to be increasing. Is it also bounded above?
• Say \( Q(m) \) means "\( a_m \leq 100 \)". Clearly true for \( m = 1 \). Suppose true for \( m < n \). Then \( a_n = 4 \sqrt{a_{n-1}} - 3 < 4 \sqrt{100} - 3 = 87 < 100 \). By induction, \( Q(n) \) holds for every \( n \), i.e. \( \{a_n\} \) is bounded above.

• Let \( Q(n) \) stand for "\( a_n < a_{n+1} \)". True for \( m = 1 \) (4 < 5). Suppose true for \( m < n \), in particular \( a_{n-1} < a_n \). So then \( \sqrt{a_{n-1}} < \sqrt{a_n} \Rightarrow a_n = 4 \sqrt{a_{n-1}} - 3 < 4 \sqrt{a_n} - 3 = a_{n+1} \). By induction, \( Q(n) \) holds for all \( n \), i.e. \( \{a_n\} \) is increasing.

• By the Monotonic Sequence Theorem, \( L = \lim_{n \to \infty} a_n \) exists. Take limits on both sides of \( a_{n+1} = 4 \sqrt{a_n} - 3 \) to obtain \( L = 4 \sqrt{L} - 3 \Rightarrow L - 4 \sqrt{L} + 3 = 0 \Rightarrow L = 1 \) or 9. Which is it? Well, \( \{a_n\} \) starts from 4 and \( \{a_n\} \) is increasing, so can't limit to 1. So \( L = 9 \).

\[ \text{Ex 2/} \quad \text{Find } L = \lim_{n \to \infty} \left(1 + \frac{a}{n}\right)^n. \]

This is an indeterminate form of type \( 1^\infty \). Take \( \ln \) of both sides, then apply L'Hopital.

\[
\ln(L) = \lim_{n \to \infty} \ln\left(\left(1 + \frac{a}{n}\right)^n\right) = \lim_{n \to \infty} n \cdot \ln\left(1 + \frac{a}{n}\right)
\]

\[
= \lim_{n \to \infty} \frac{\ln(1 + \frac{a}{n})}{\frac{1}{n}} = \lim_{n \to \infty} \frac{\frac{1}{1 + \frac{a}{n} \cdot \frac{n}{a}} \cdot \frac{a}{n}}{\frac{1}{n}} \quad \text{[L'Hopital]} \\
= \lim_{x \to \infty} \frac{a}{1 + \frac{a}{x} \cdot \frac{x}{a}} = \frac{a}{1 + \lim_{x \to \infty} \frac{a}{x}} = \frac{a}{1 + 0} = a.
\]

\[ \Rightarrow \quad L = e^{\ln(L)} = e^a. \]

Turning to series:

\[ \text{Ex 3/} \quad \text{Compute } \sum_{n=1}^{\infty} \frac{3^n}{5^{n-1}}. \]
This is a geometric series with ratio \( r = \frac{a_{n+1}}{a_n} = \frac{3^{n+1}}{5^{n+1}} \cdot \frac{5^n}{3^n} = \frac{3}{5} \) and initial term \( 3^{1/5} \cdot 5^{-1} = 9 \). Since \(-1 < \frac{3}{5} < 1\), the series converges, with sum \( \frac{9}{1 - \frac{3}{5}} = \frac{9}{\frac{2}{5}} = \frac{45}{2} \).

**Ex 4/ Which of \( a = 0, 1, 2, 3 \) gives the tightest but still correct inequality?**

\[ \sum_{n=2}^{\infty} n^2 e^{-n^3} < \int_{a}^{\infty} x^2 e^{-x^3} \, dx. \]

The integrand is decreasing (by taking derivatives) for \( x > \frac{3^3}{3} \). So the picture is as shown, with \( a = 0 \) or \( 1 \) giving a correct inequality. On the other hand, if we look at \( a = 2 \) the inequality goes the other way, as you can see from this modification of the picture: the area under the curve is contained in the area under the boxes.

We also might ask: does the series converge?

Compute the integral:

\[ \int_{1}^{\infty} x^2 e^{-x^3} \, dx = \lim_{b \to \infty} \int_{1}^{b} x^2 e^{-x^3} \, dx = \lim_{b \to \infty} \left[ -\frac{1}{3} e^{-x^3} \right]_{1}^{b} = \lim_{b \to \infty} \left( \frac{1}{3} e^{-b^3} - \frac{1}{3} e^{-1} \right) = \frac{1}{3e}. \]

(How do I know that \( e^{-b^3} \to 0 \) ? It is \((e^{-1}) b^3\), where \( e^{-1} < 1 \) and \( b^3 \to \infty \).)

I'd also recommend looking over the past Participar problems on sequences & series.
For improper integrals, the trickier ones — though far less common — are the ones with a vertical asymptote.

**Ex 5**/ Determine which of \( \int_{-2}^{1} \frac{dx}{x^2} \) and \( \int_{0}^{3} \frac{dx}{(x-1)^{3/2}} \) converge, and compute its value.

- \( \int_{-2}^{1} \frac{dx}{x^2} = \lim_{b \to 0^-} \int_{-2}^{b} \frac{dx}{x^2} + \lim_{b \to 0^+} \int_{b}^{1} \frac{dx}{x^2} \)
  \( = \lim_{b \to 0^-} \left( \frac{-1}{x} \right)_{-2}^{b} + \lim_{b \to 0^+} \left( \frac{-1}{x} \right)_{b}^{1} \)
  \( = \lim_{b \to 0^-} \left( \frac{-1}{b} - \frac{-1}{-2} \right) + \lim_{b \to 0^+} \left( \frac{-1}{1} - \frac{-1}{b} \right) \)
  \( = \text{diverges} \)

(If you ignore the asymptote, you'd get \( \frac{-1}{1} - \frac{-1}{-2} = \frac{-3}{2} \), which is absurd)

- \( \int_{0}^{3} \frac{dx}{(x-1)^{3/2}} = \lim_{b \to 1^-} \int_{0}^{b} (x-1)^{-3/2} dx + \lim_{b \to 1^+} \int_{b}^{3} (x-1)^{-3/2} dx \)
  \( = \lim_{b \to 1^-} \left. \frac{-2}{(x-1)^{1/2}} \right|_{0}^{b} + \lim_{b \to 1^+} \left. \frac{-2}{(x-1)^{1/2}} \right|_{b}^{3} \)
  \( = -3(1)^{1/2} + 3(2)^{1/2} = 3(\sqrt{2} + 1). \)

Arclength and surface area are more straightforward:

\[
\text{Arc length: } l = \int_{a}^{b} \sqrt{1 + (f'(x))^2} \, dx
\]

\[
\text{Surface area: } A = \int_{a}^{b} \frac{2\pi f(x)}{\sqrt{1 + (f'(x))^2}} \, dx
\]

Probability: a density function \( f(x) \) is \( \geq 0 \) and satisfies \( \int_{-\infty}^{\infty} f(x) \, dx = 1 \). (Sometimes you’re asked to determine a constant to make this so.) The mean \( \mu = \int_{-\infty}^{\infty} x f(x) \, dx \), and the median \( m \) satisfies \( \frac{1}{2} = \int_{m}^{\infty} f(x) \, dx \). That’s it.