Lecture 34: Alternating series

The integral and comparison tests described last week were for series $\sum a_n$ with all $a_n \geq 0$. Today we will look at an important example of series with both positive and negative terms.

$\boxed{\text{Definition}}$ An <u>alternating series</u> is a series $\sum_{n\geq 1} a_n$ whose terms have sign alternating between + & −. This looks like

$$\sum_{n\geq 1} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \dots$$

or

$$\sum_{n\geq 1} (-1)^n b_n = -b_1 + b_2 - b_3 + b_4 - \dots$$

with all $b_n \geq 0$.

For instance, we know that the harmonic series $\sum_{n\geq 1} \frac{1}{n}$ diverges. Is the same true of the <u>alternating harmonic series</u>

$$\sum_{n\geq 1} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots \quad ?$$

Suppose that $\{b_n\}$ is decreasing: i.e. $b_n > b_{n+1}$ for all n.

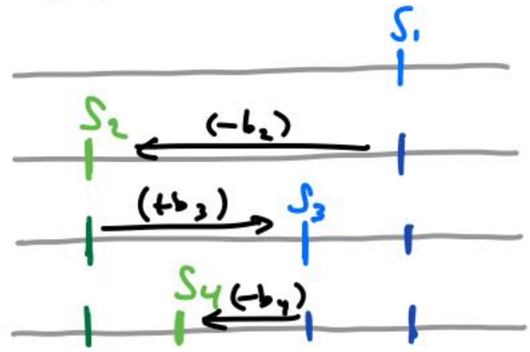
The partial sums of $\sum (-1)^{n+1} b_n$ are

$S_1 = b_1$

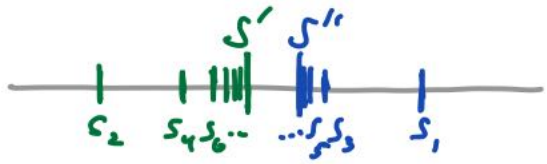
$S_2 = b_1 - b_2 = S_1 - b_2$

$S_3 = b_1 - b_2 + b_3 = S_2 + b_3$

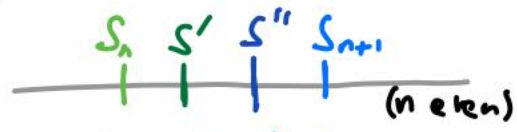
$S_4 = b_1 - b_2 + b_3 - b_4 = S_3 - b_4$



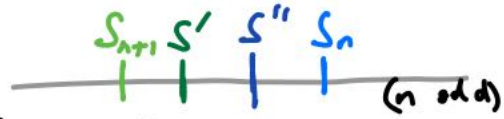
Continuing on, we arrive at the picture:



in which the even-numbered partial sums are increasing and bounded above, hence converge to some limit S'; and where the odd-numbered terms are decreasing and bounded below, hence converge to some limit S''.



Both S' and S'' are between S_n and S_{n+1} for all n, and so



$|S'' - S'| \le |S_{n+1} - S_n| = b_{n+1}$ (for any n).

If $\lim_{n\to\infty} b_n = 0$, then this forces $S'' = S'$, which means that the sequences of odd & even partial sums have a common limit $S = \lim_{n\to\infty} S_n$. Furthermore, since S lies between S_n and S_{n+1},

$$|S - S_n| \le |S_{n+1} - S_n| = b_{n+1}.$$

So we arrive at the

$\boxed{\text{Alternating Series Test}}$ Given an alternating series $\sum_{n\ge 1} (-1)^{n+1} b_n$, with b_n a <u>decreasing</u>* sequence of positive numbers, <u>limiting to 0</u>. Then the series converges to a sum S, and the error $|S - S_n|$ is not more than the next term b_{n+1} in the sequence.

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* $\{b_n\}$ need only be "eventually decreasing", i.e. $b_{n+1} < b_n$ for all $n \ge N$ (for some N).

Ex 1/ $\sum_{n\geq 1}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ Conv./div.?

Converges, since it satisfies the hypotheses of the AST.

How many terms must we sum to get within 0.01 of S?

We want $|S - S_n| \leq 0.01$. This holds if $b_{n+1} \leq 0.01$,

i.e. $\frac{1}{n+1} \leq 0.01 \iff n+1 \geq 100 \iff n \geq 99$. //

So, what is the sum? Notice that

$$S_{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}$$

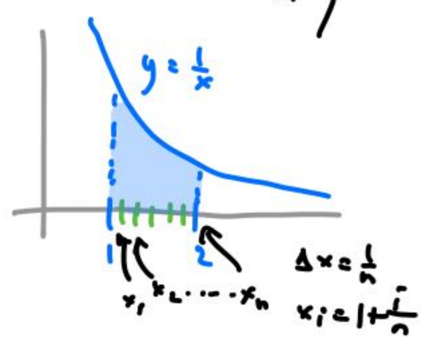
$$- \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$$

$$= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$

$$= \frac{1}{n}\left(\frac{n}{n+1} + \frac{n}{n+2} + \cdots + \frac{n}{2n}\right)$$

$$= \frac{1}{n}\left(\frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \cdots + \frac{1}{1+\frac{n}{n}}\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n} f(x_i), \text{ where } f(x) = \frac{1}{x}.$$



That is, S_{2n} is a Riemann sum for the area depicted,

and
$$S = \lim_{n\to\infty} S_{2n} = \int_1^2 \frac{dx}{x} = \ln(2).$$

Ex 2/ Does $\sum_{n=1}^{\infty}(-1)^{n+1}\underbrace{\frac{n}{n^2+1}}_{b_n}$ converge?

- Is it true that $b_{n+1} \leq b_n$? i.e.

$$\frac{n+1}{(n+1)^2+1} \le \frac{n}{n^2+1}$$

$$(n^2+1)(n+1) \le ((n+1)^2+1)\,n$$

$$n^3+n^2+n+1 \le n^3+2n^2+2n$$

$$0 \le n^2+n-1 \quad \leftarrow \text{true for } n \ge 1.$$

- Is it true that $\lim_{n\to\infty} b_n = 0$? Yes (apply L'Hopital, or write $\lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{n/n^2}{n^2/n^2+1/n^2} = \lim_{n\to\infty} \frac{n^{-1}}{1+n^{-2}} = \frac{0}{1} = 0.$)

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Ex 3/ What about $\sum_{n\ge 1} (-1)^n \frac{3n+5}{n+1}$?

- $b_{n+1} \le b_n \iff \frac{3n+8}{n+2} \le \frac{3n+5}{n+1} \iff 3n^2+11n+8 \le 3n^2+11n+10$ which is true.
- $\lim_{n\to\infty} b_n = 3$, not 0. Series $\therefore$ diverges by the "test for divergence".

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Ex 4/ How about $1 - \frac{1}{4} + \frac{1}{3} - \frac{1}{16} + \frac{1}{5} - \frac{1}{36} + \frac{1}{7} - \frac{1}{64} + \frac{1}{9} - \dots$?

- Here $b_n = \frac{1}{n}$ for n odd and $\frac{1}{n^2}$ for n even.

  So $\lim_{n\to\infty} b_n = 0$.
- __BUT__ b_n is not decreasing!! It's basically $\sum_{m\ge 1}\left(\frac{1}{2m-1} - \frac{1}{(2m)^2}\right)$

  $= \sum_{m\ge 1} \frac{4m^2-2m+1}{(2m)^2(2m-1)}$, which diverges by the limit comparison test.

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Ex 5/ One more: $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{2^n}$.

- $b_n = \frac{n^2}{2^n}$: we do have $\lim_{n\to\infty} b_n = \lim_{x\to\infty} \frac{x^2}{2^x} \underset{\uparrow \text{L'hôpital}}{=} \lim_{x\to\infty} \frac{2x}{2^x \ln 2}$

  $\underset{\uparrow \text{L'hôpital}}{=} \lim_{x\to\infty} \frac{2}{2^x (\ln 2)^2} = 0$.

- **BUT** : $b_1 = \frac{1}{2}$, $b_2 = 1$, $b_3 = \frac{9}{8}$, . . . not decreasing?

  Look at $f(x) = \frac{x^2}{2^x}$. The derivative is

  $$f'(x) = \frac{2^x \cdot 2x - x^2 2^x \ln 2}{2^{2x}} = \frac{x\, 2^x (2 - x \ln 2)}{2^{2x}}$$

  $$\approx \frac{x}{2^x}(2 - 0.69x) \quad \text{which is } < 0 \text{ for } x \geq 3 .$$

  So $\frac{n^2}{2^n}$ is decreasing for $n \geq 3$: $b_4 = 1$, $b_5 = \frac{25}{32}$, $b_6 = \frac{36}{64}$ and so on.

$\Rightarrow$ by Alternating Series Test, the series converges. //